CS103, Autumn 2026

Lecture 13:

Mathematical Induction

Part 2 of 2

Mathematical Induction

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1. Recap from Last Time
2. The Colored Cubes Problem
3. “Building Up” vs. “Building Down”
4. Announcements
5. Complete Induction
6. Eating a Chocolate Bar
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Let P be some predicate.

$$\forall n \in \mathbb{N}.\ P(n)$$

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true…

$$P(0) \text{ is true}$$

…and it stays true…

and

$$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$$

then

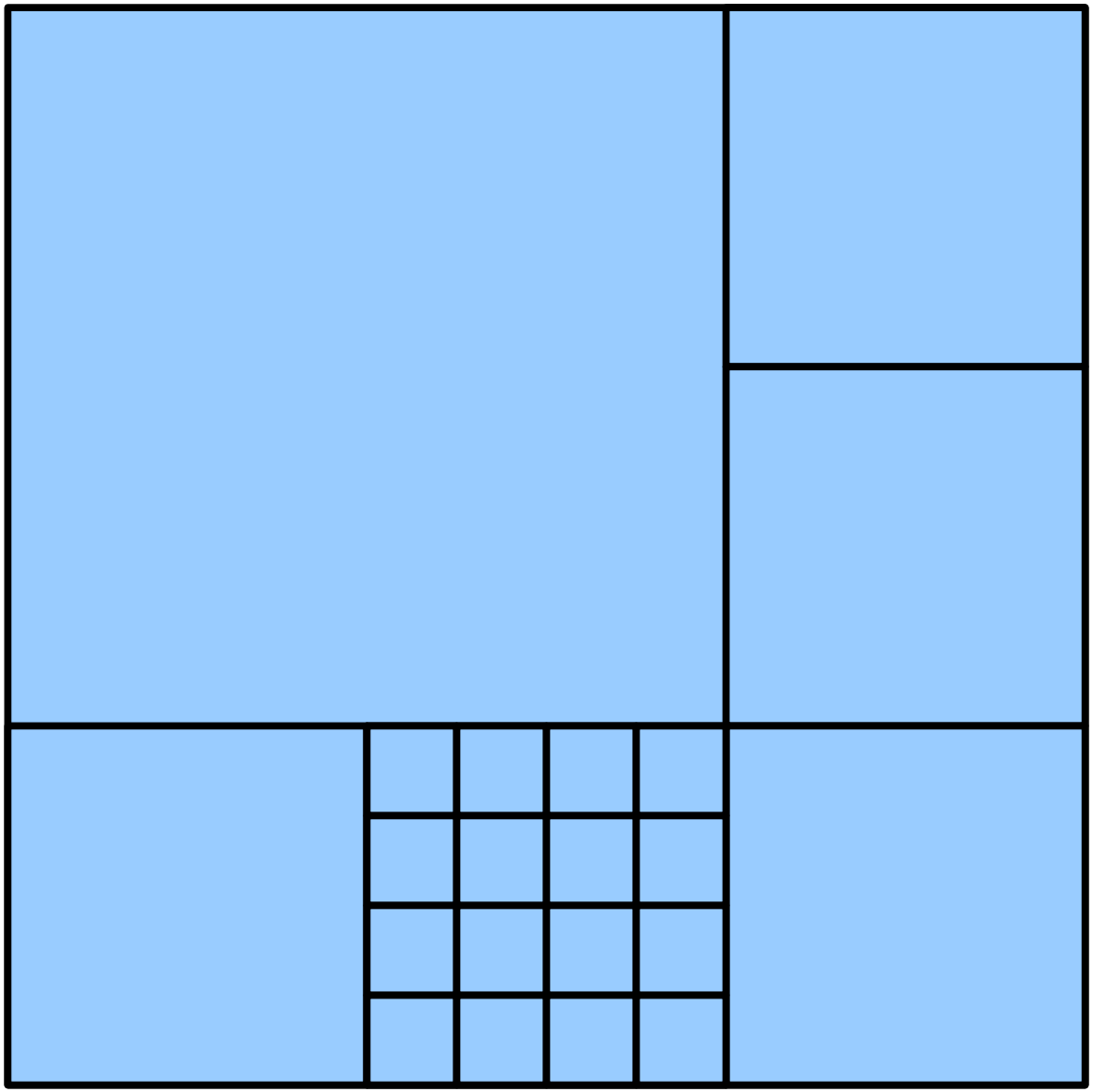
$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

Variations on Induction

For what values of n can a square be subdivided into n squares?

Subdividing a Square



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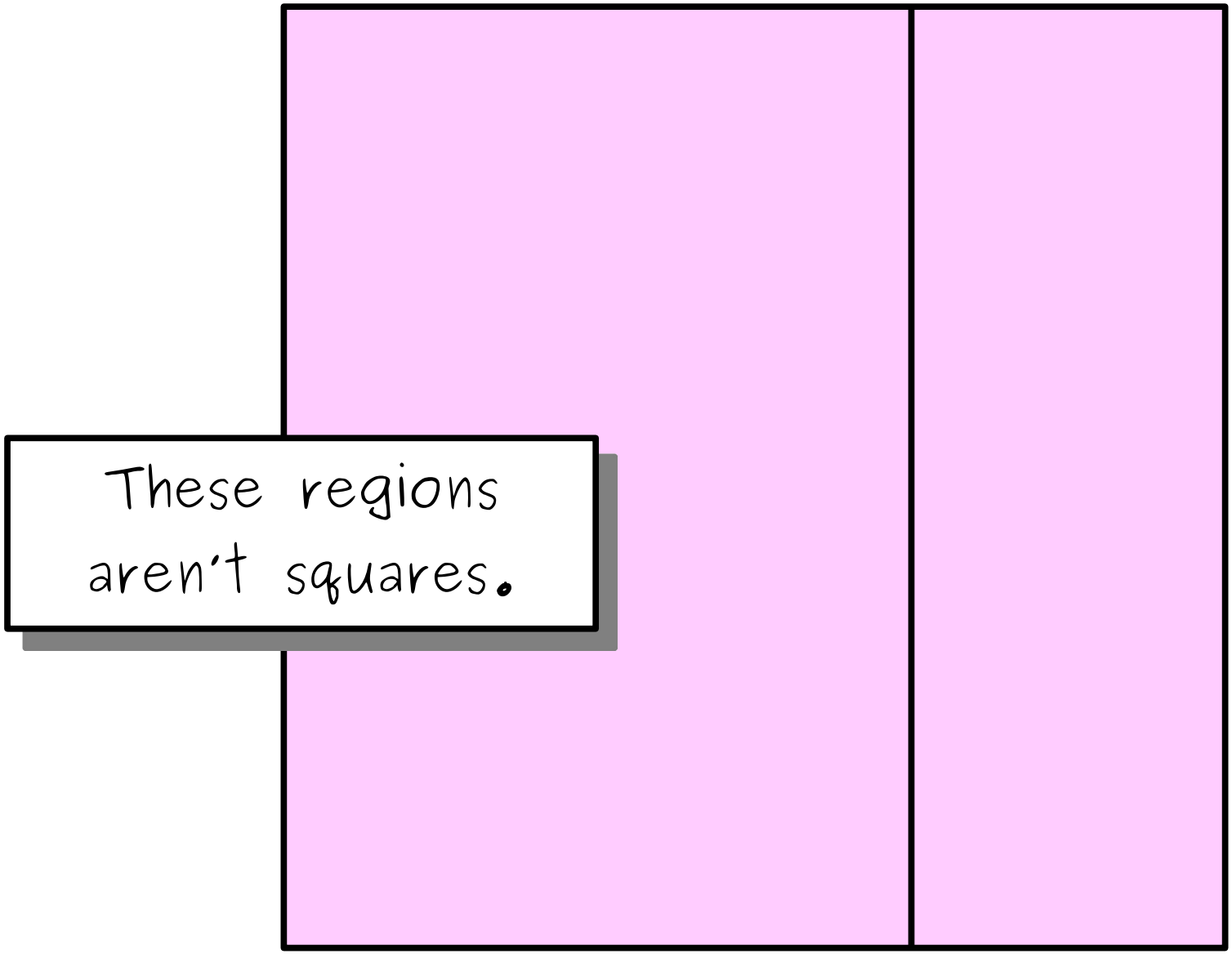

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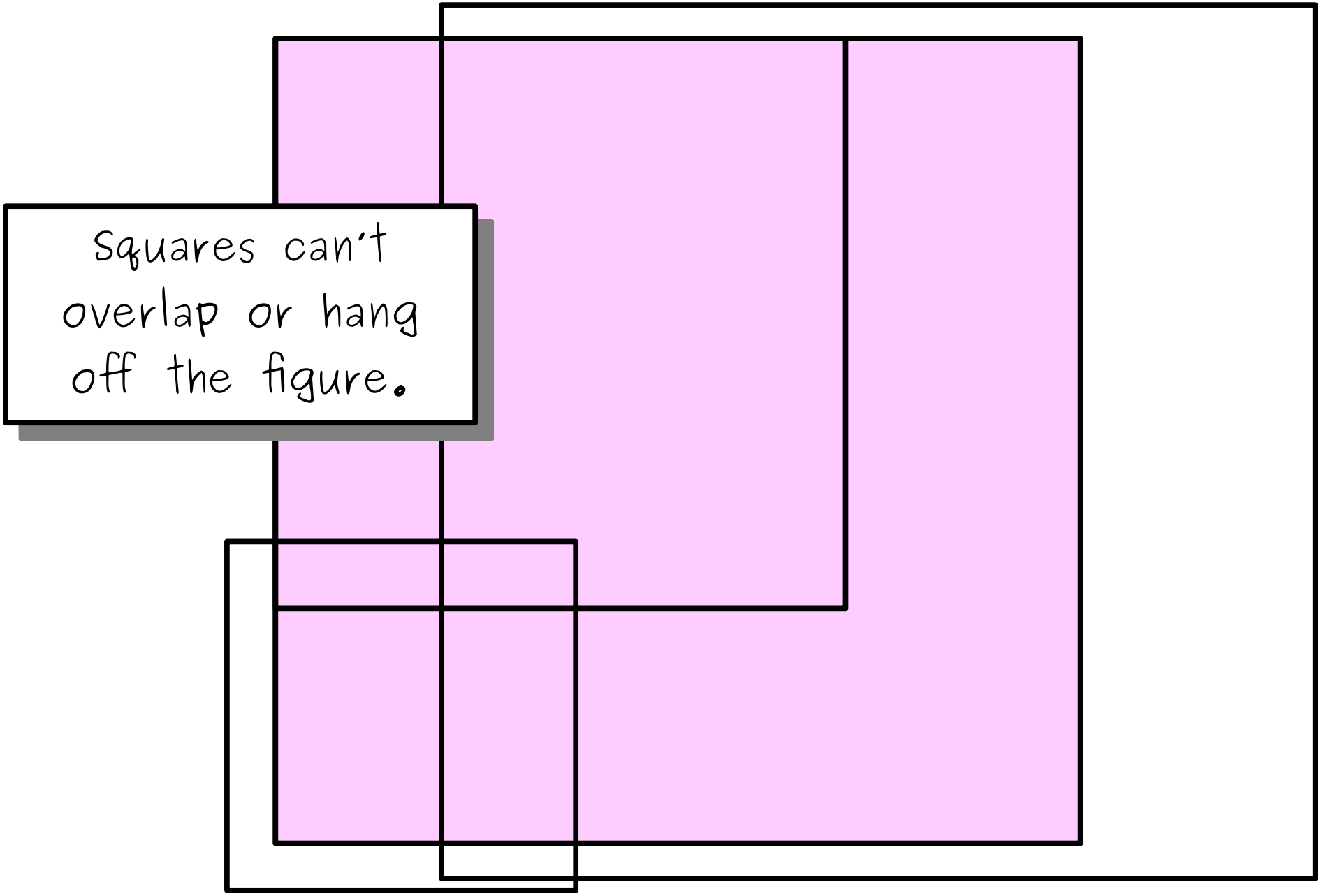
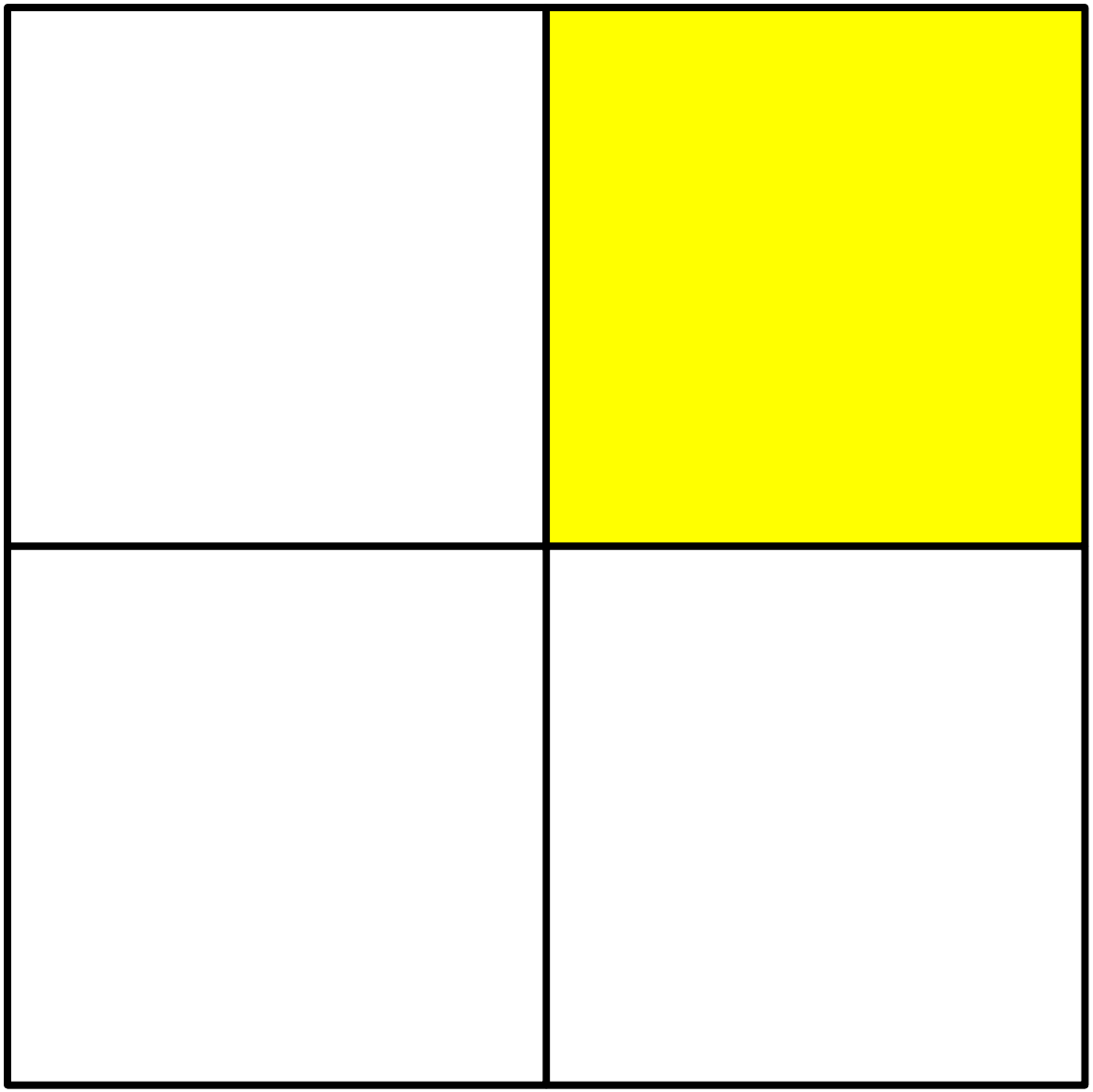


An Insight



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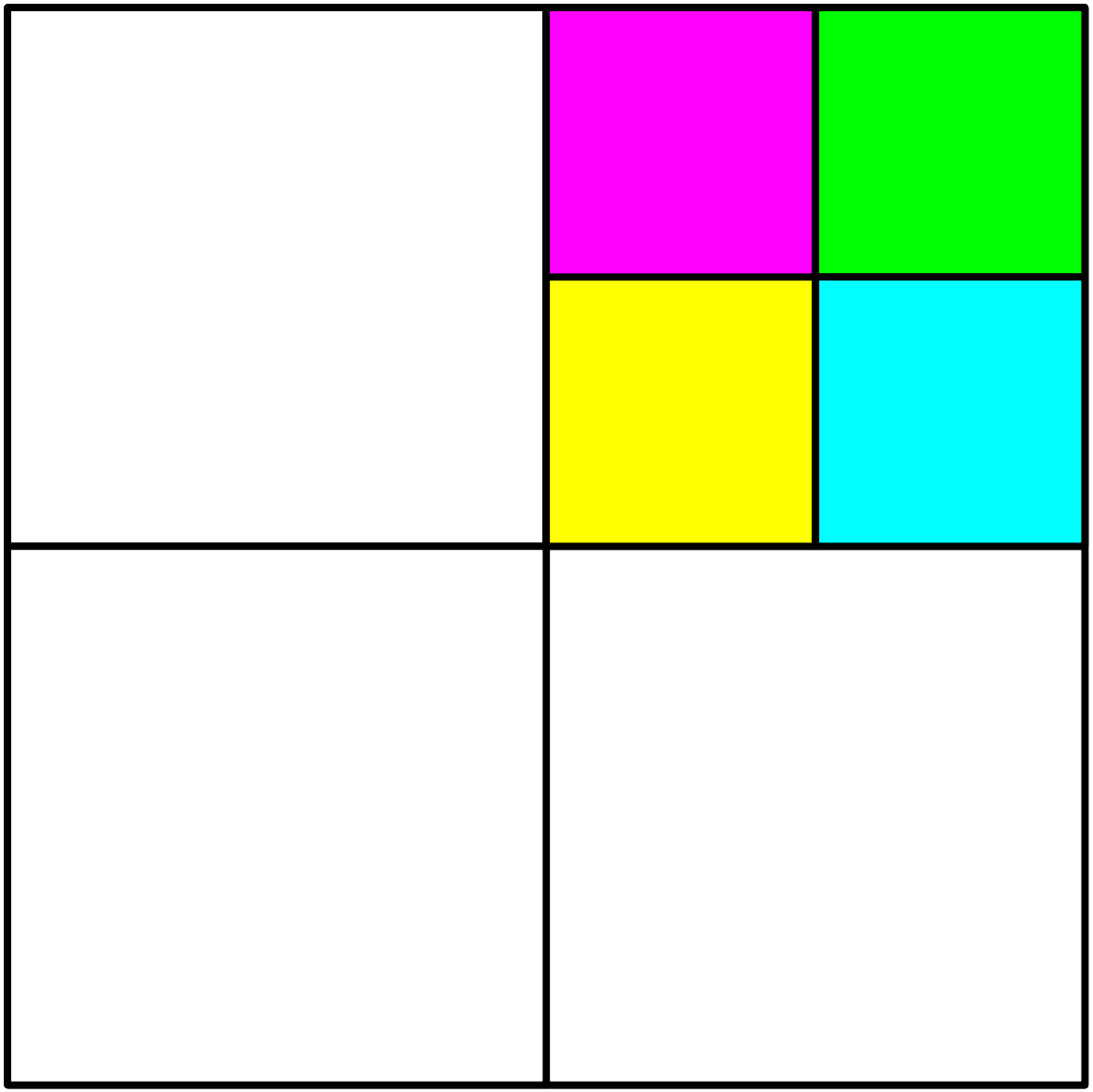

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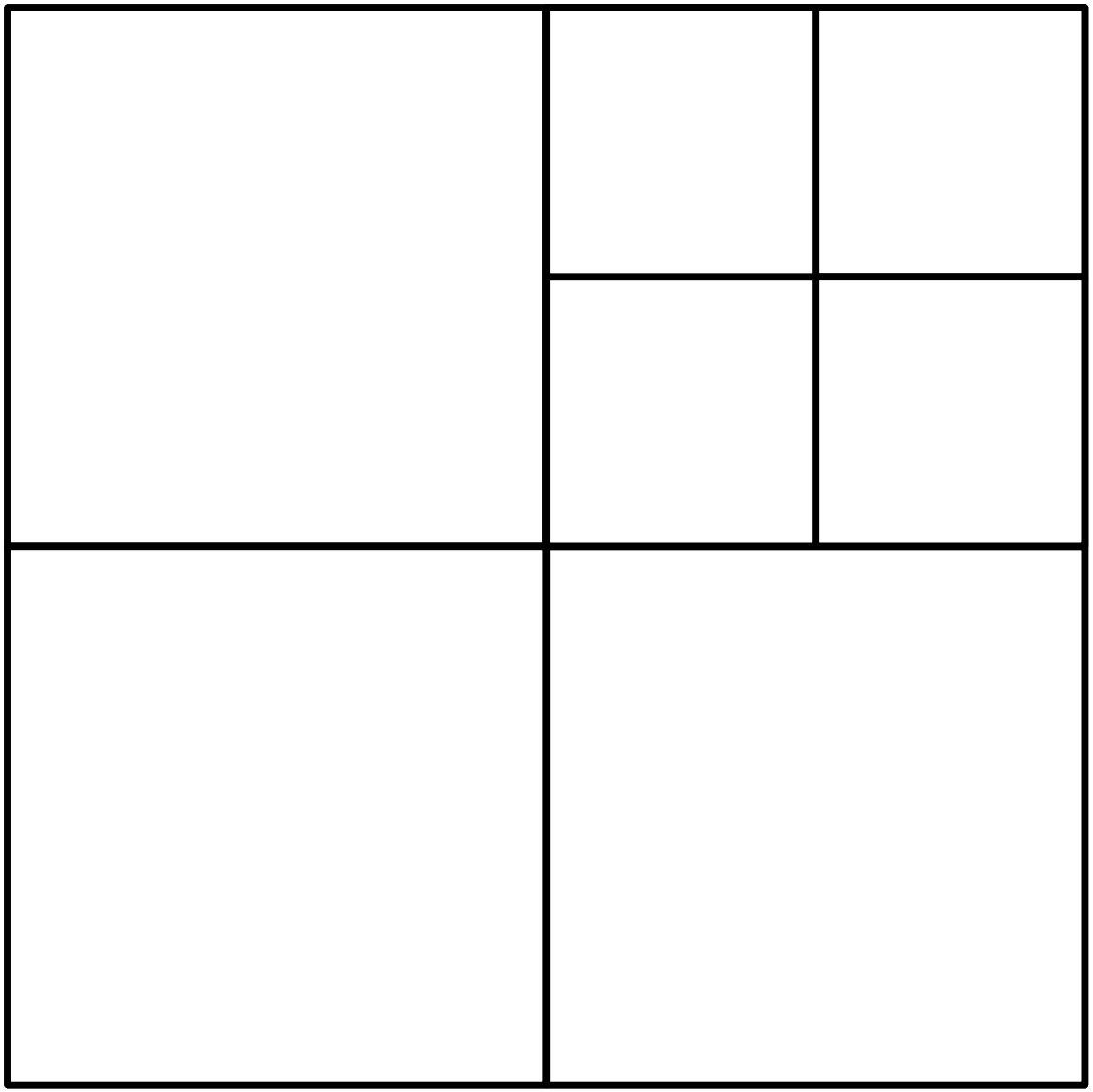

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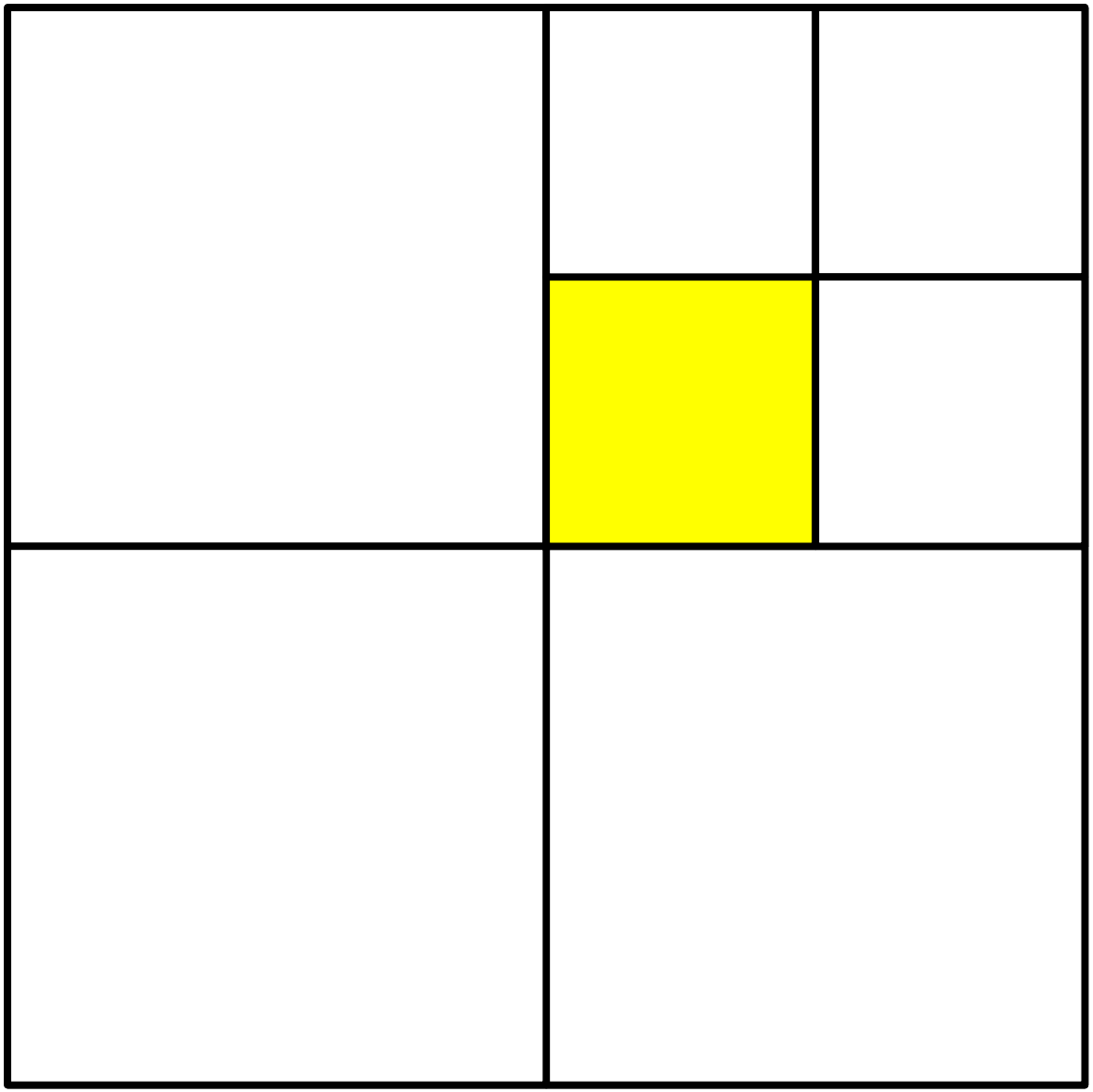

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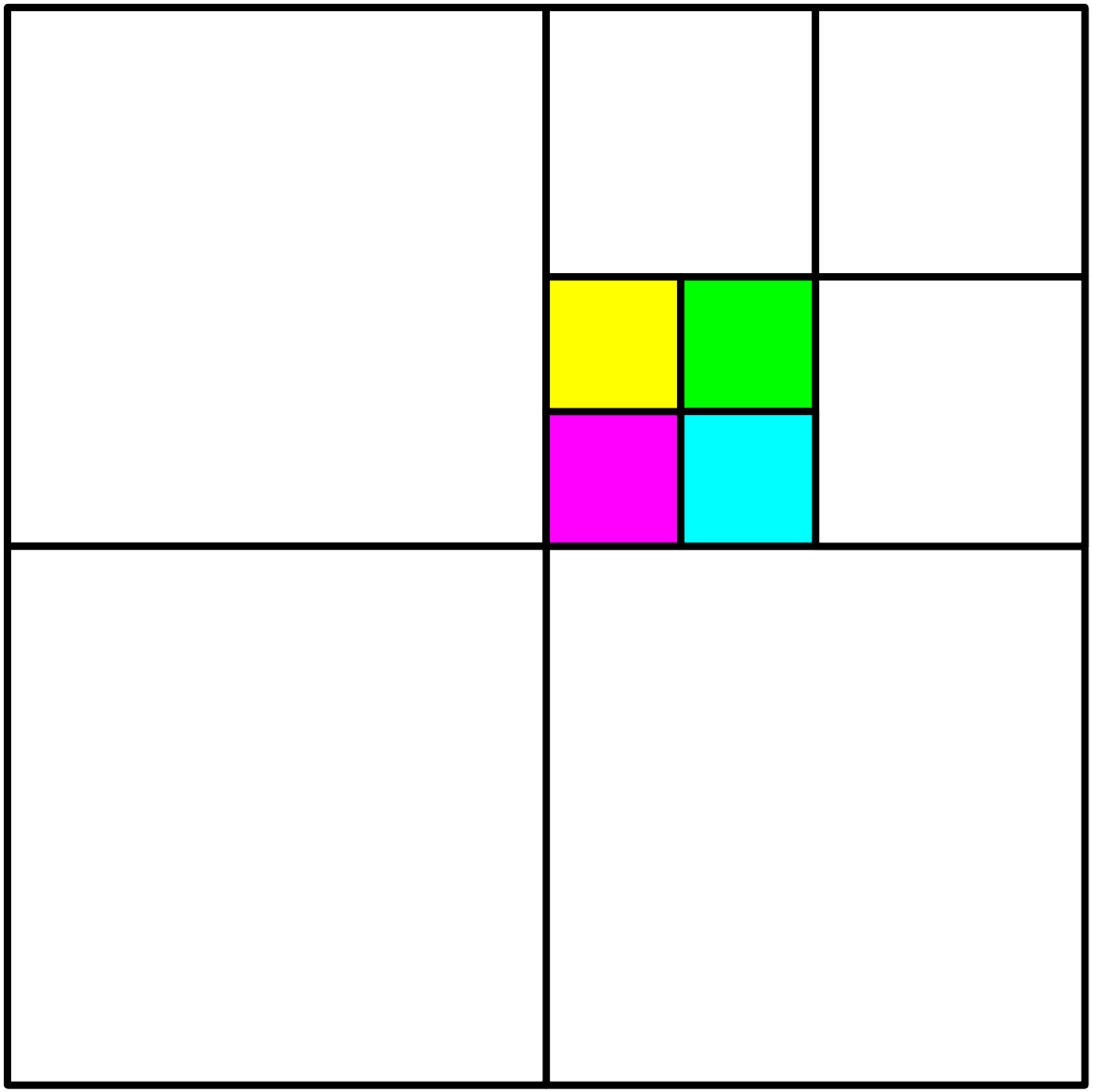

Theorem: For any $n \geq 6$, there is a way to subdivide a square into n smaller squares.

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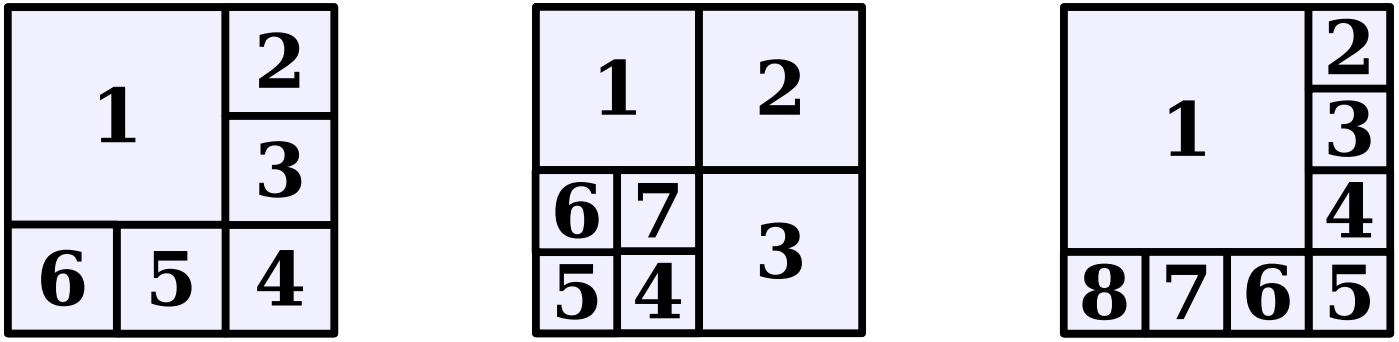
What mathematical object do we need in order to kick things off?

Theorem: For any $n \geq 6$, there is a way to subdivide a square into n smaller squares.

Proof: Let $P(n)$ be the statement "there is a way to subdivide a square into n smaller squares." We will prove by induction that $P(n)$ holds for all $n \geq 6$, from which the theorem follows.

As our base cases, we prove $P(6)$, $P(7)$, and $P(8)$, that a square can be subdivided into 6, 7, and 8 squares. This is shown here:



For the inductive step, assume that for some arbitrary $k \geq 6$ that $P(k)$ is true and that there is a way to subdivide a square into k squares. We prove $P(k+3)$, that there is a way to subdivide a square into $k+3$ squares. To see this, start by obtaining (via the inductive hypothesis) a subdivision of a square into k squares. Then, choose any of the squares and split it into four equal squares. This removes one of the k squares and adds four more, so there will be a net total of $k+3$ squares. Thus $P(k+3)$ holds, completing the induction. ■

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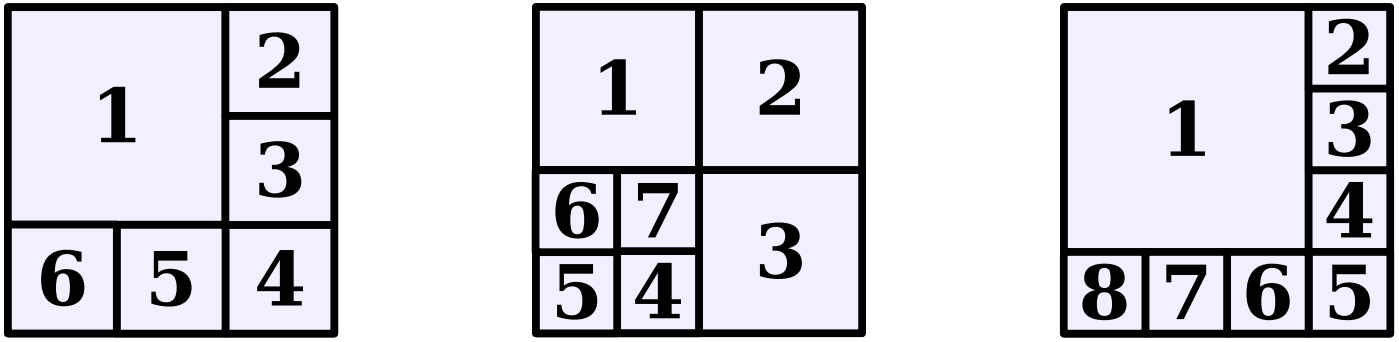

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Generalizing Induction

- When doing a proof by induction,
  - feel free to use multiple base cases, and
  - feel free to take steps of sizes other than one.
- If you do, make sure that…
  - … you actually need all your base cases. Avoid redundant base cases that are already covered by a mix of other base cases and your inductive step.
  - … you cover all the numbers you need to cover. Trace out your reasoning and make sure all the numbers you need to cover really are covered.
- As with a proof by cases, you don’t need to separately prove you’ve covered all the options. We trust you.

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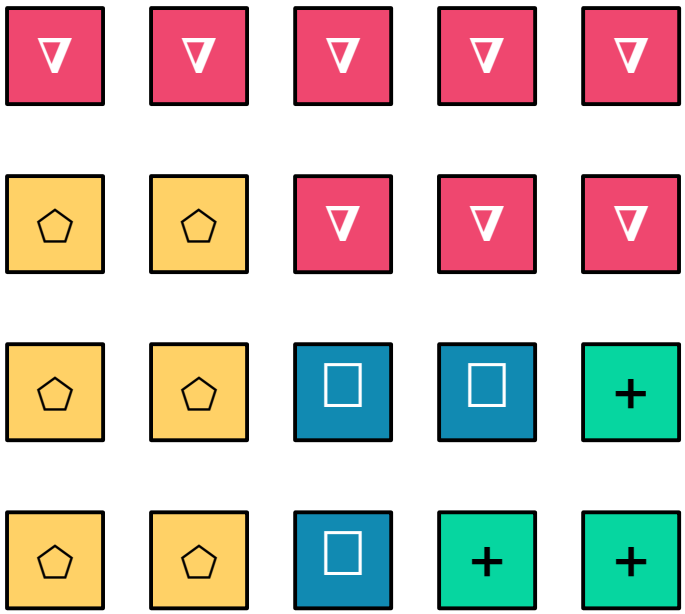

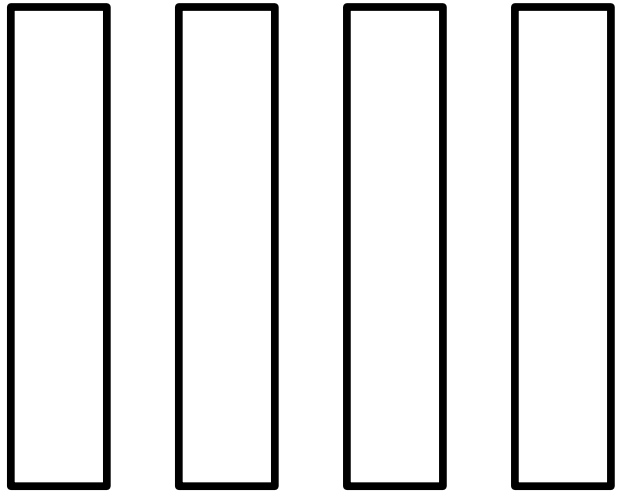

Here are 20 cubes of 4 different colors.
Split them into 4 groups of 5 cubes each so that each group has cubes of at most two different colors.

The Colored Cubes Problem

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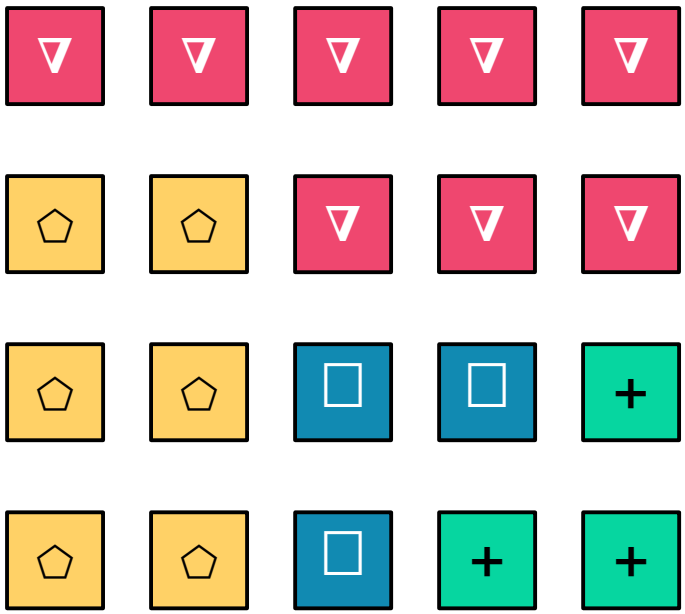
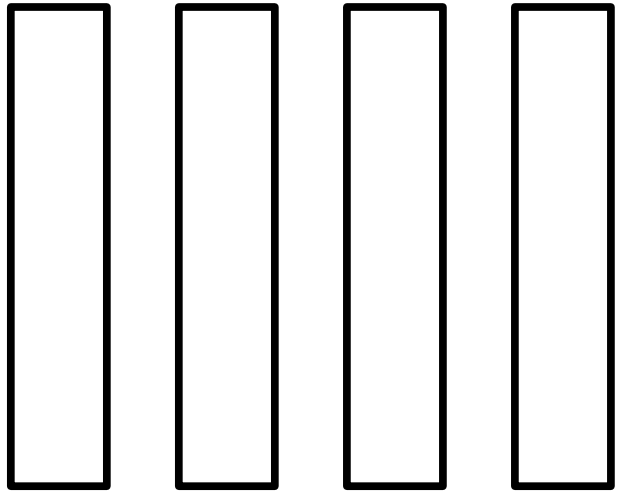
The Magic Potions Problem





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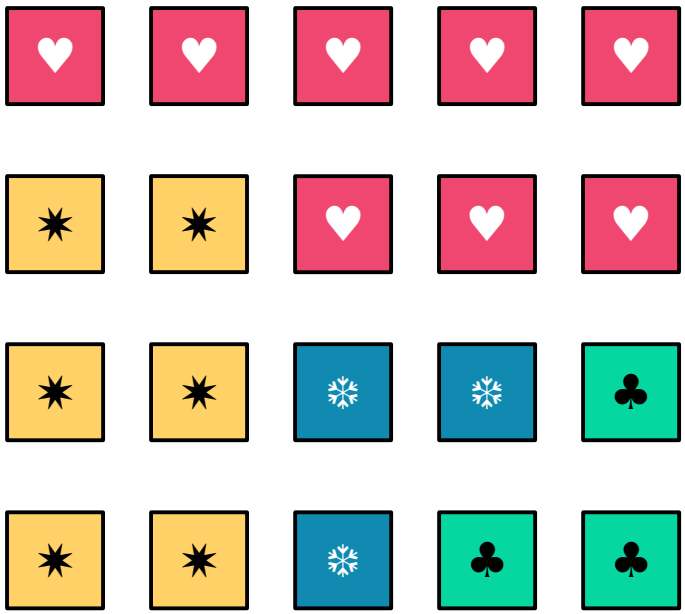

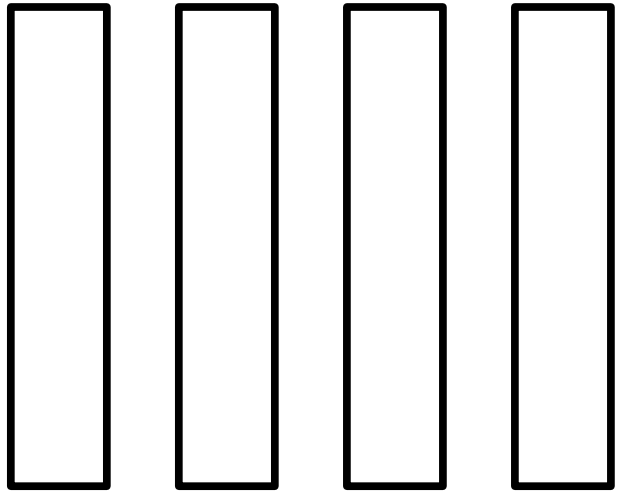

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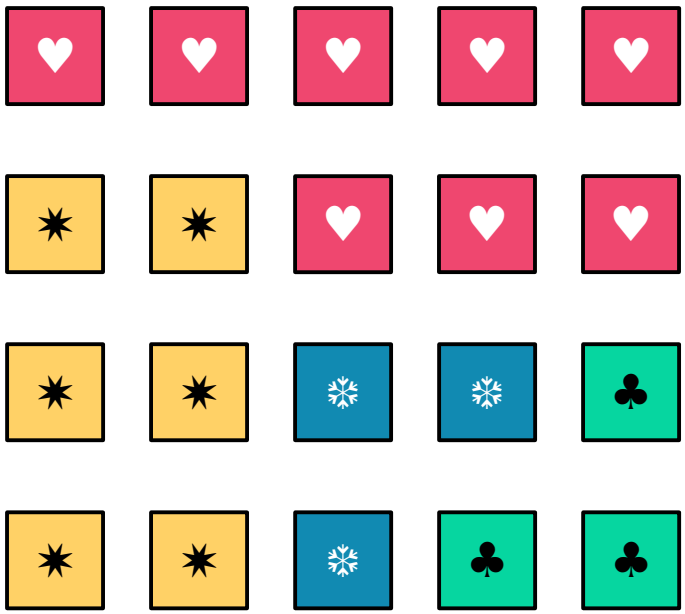

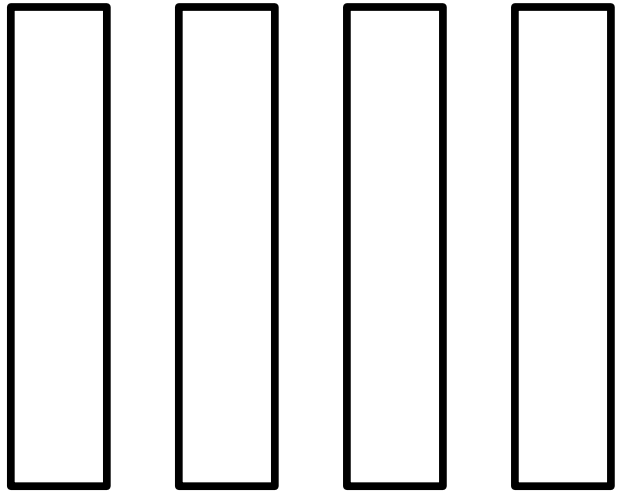

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Split them into 4 **shelves** of 5 **potions** each so that each **shelf** has **potions** of at most two different **colors**.

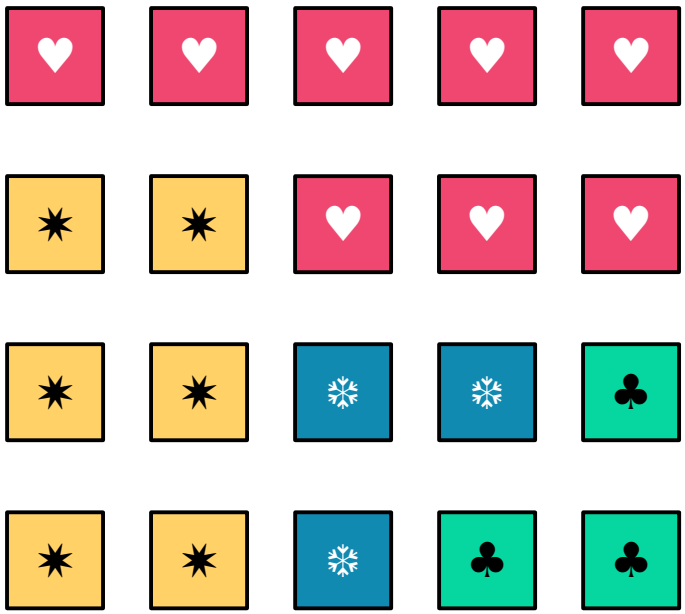
To be clear, here's the setup:

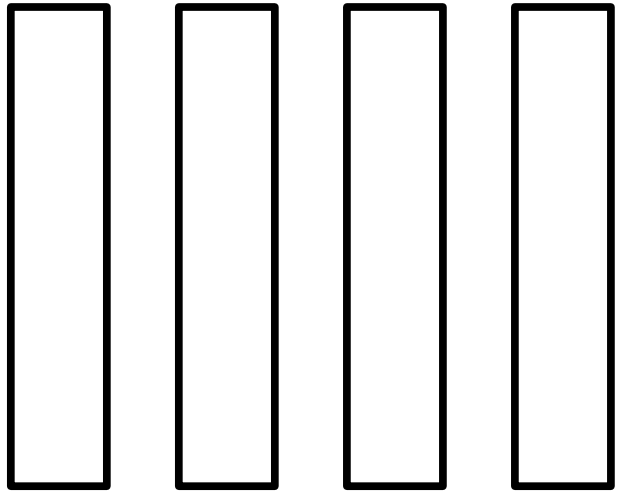
Each shelf will ***always*** have 5 potions.

A shelf will ***never*** have potions of more than 2 different colors.

We'll always have **5*n* potions** of ***n* colors** (where $n \in \mathbb{N}$).

Currently, we have **20 potions** and **4 colors** (so, $n = 4$).





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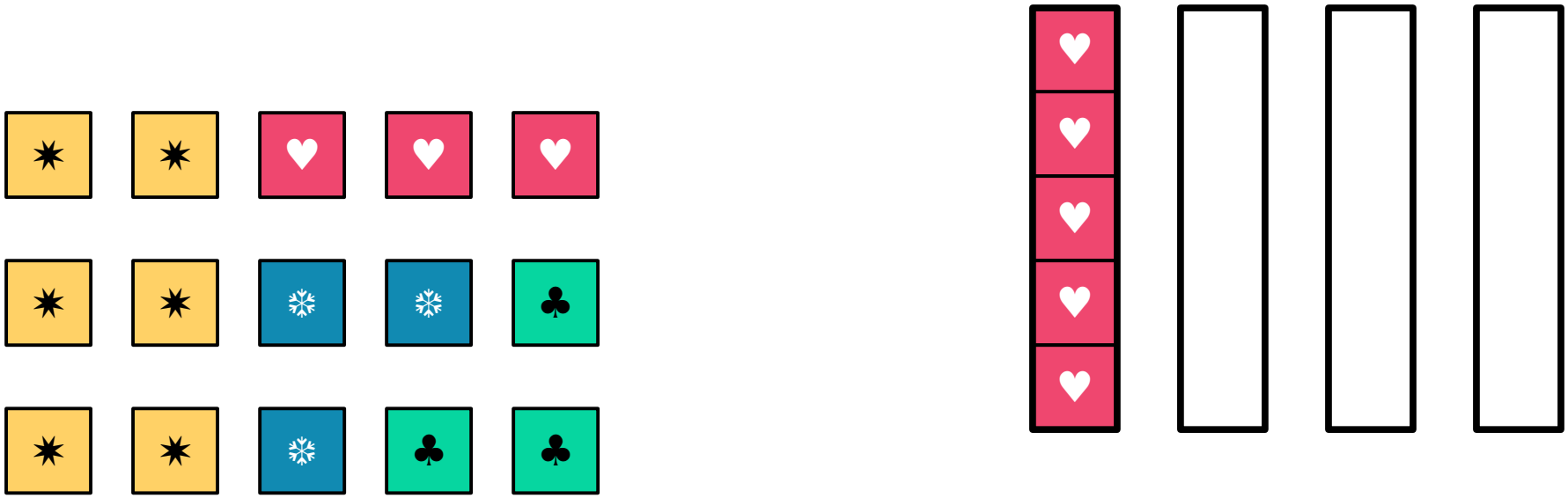

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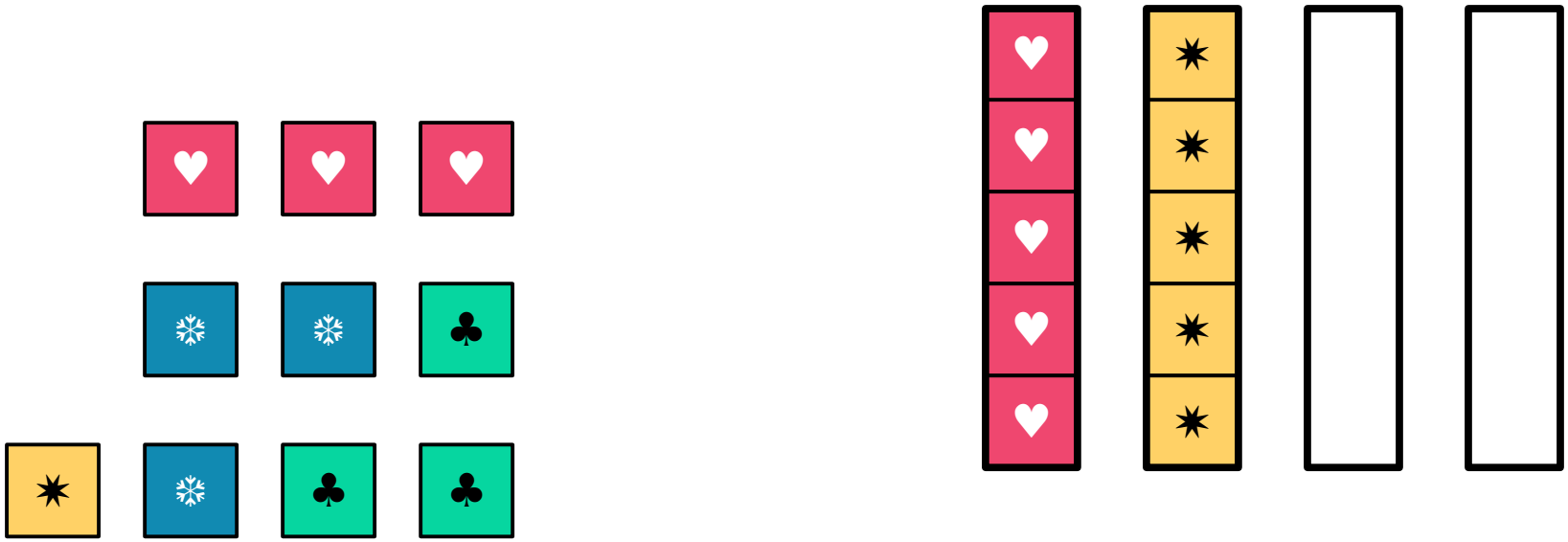

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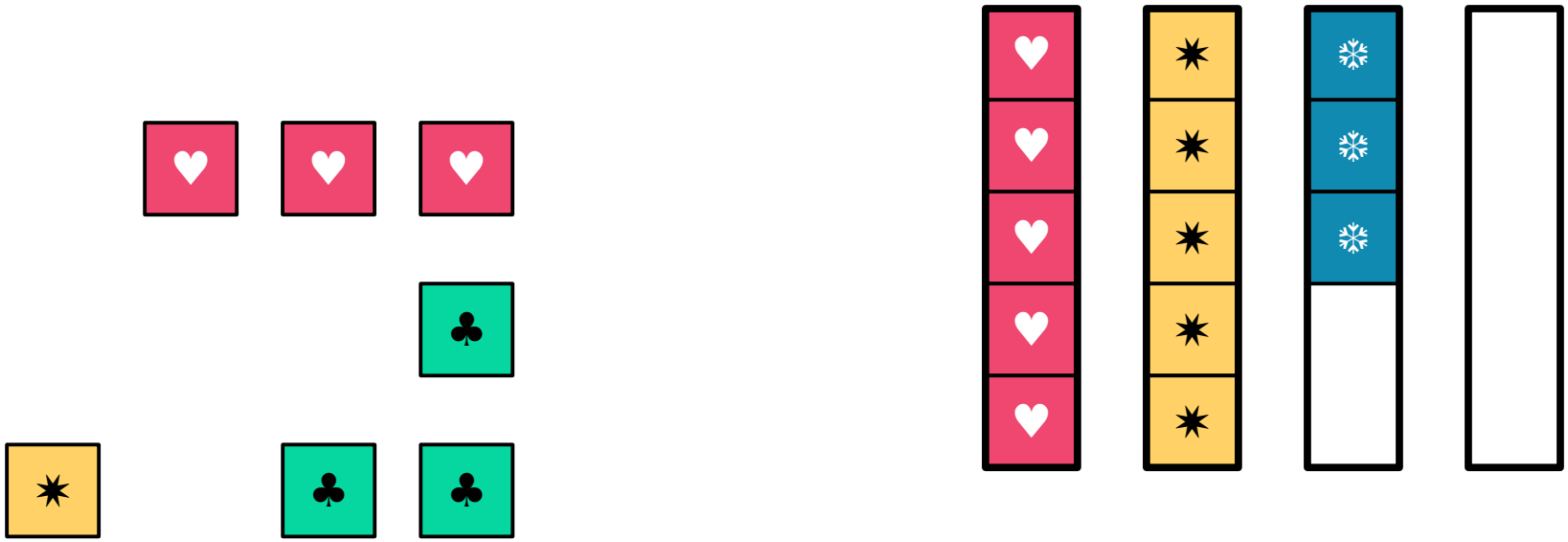

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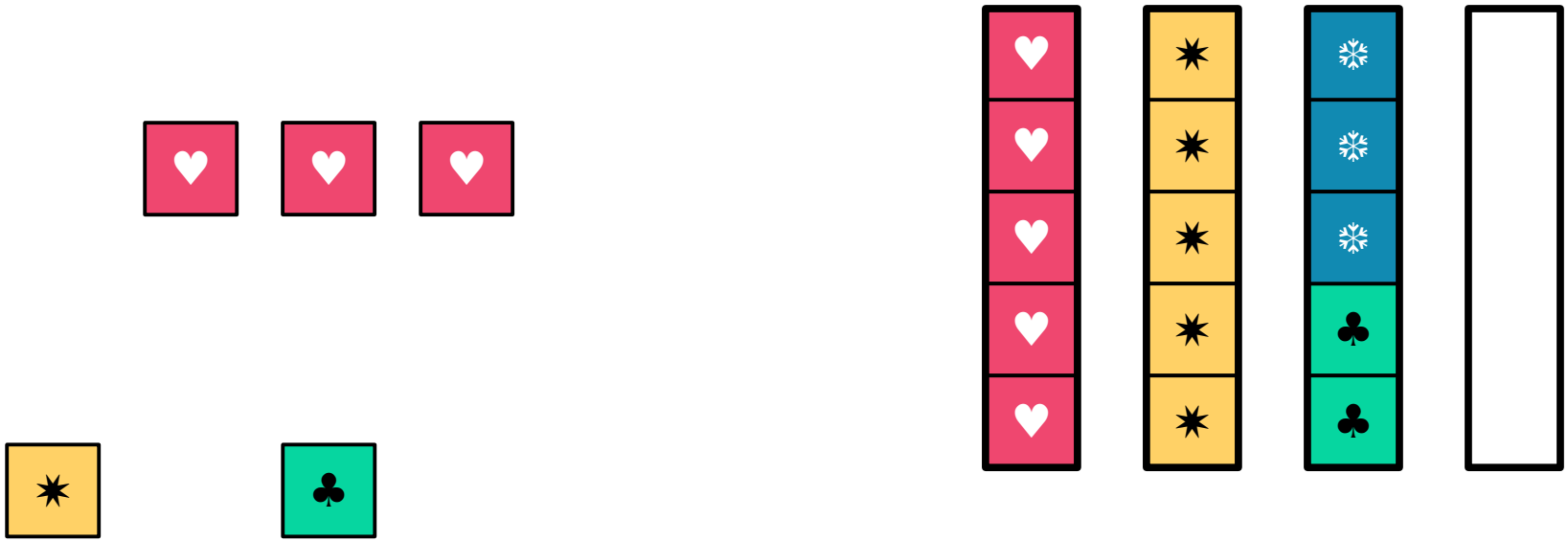

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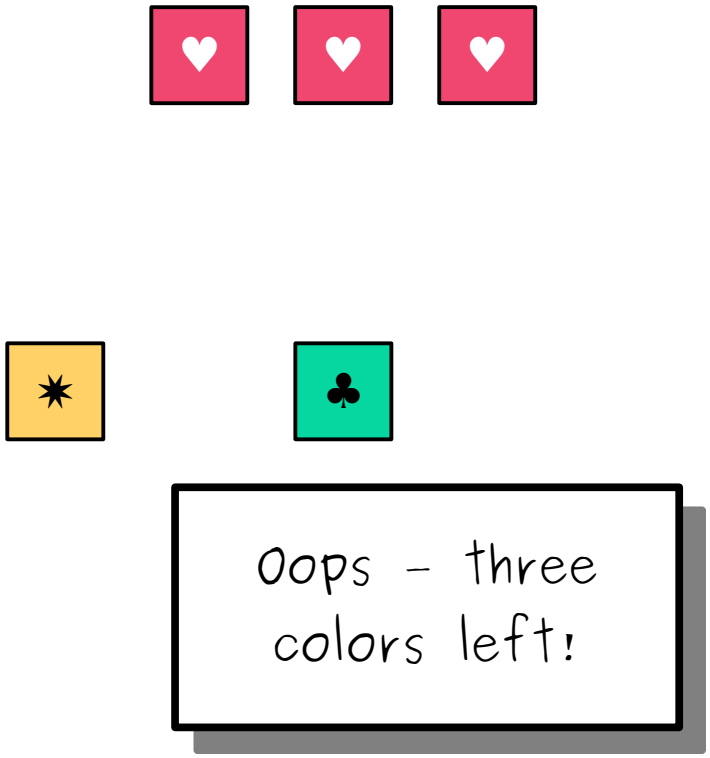

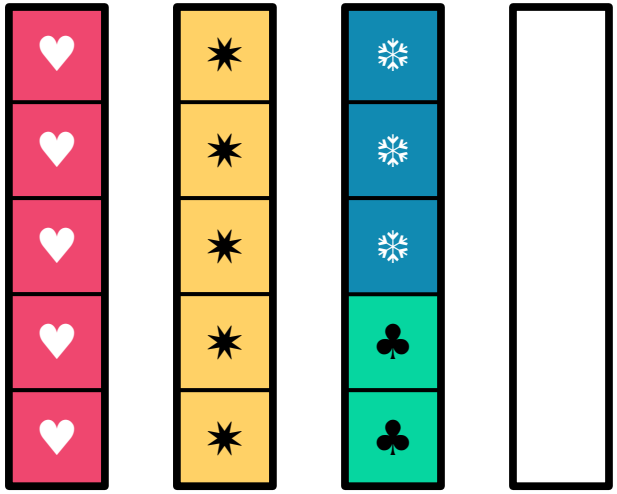

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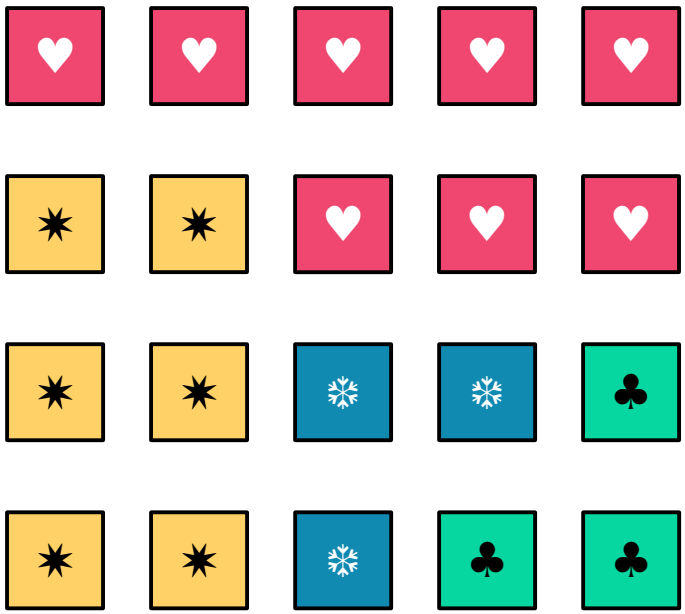

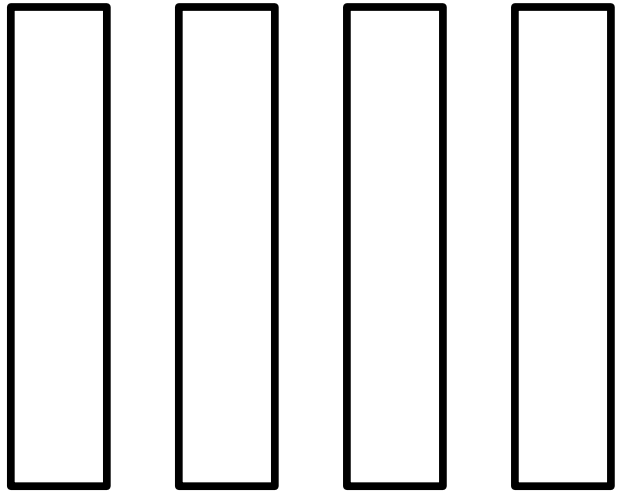

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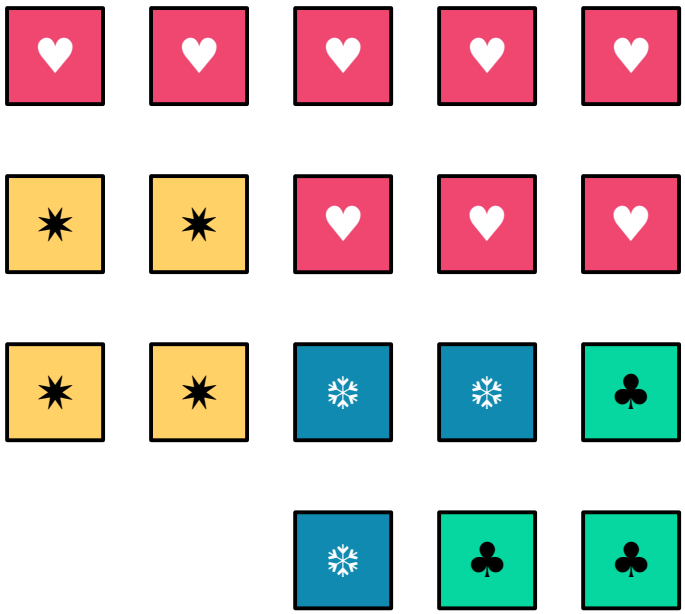

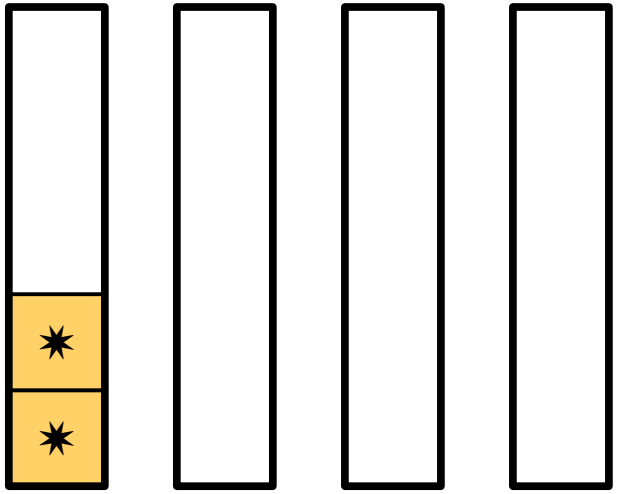

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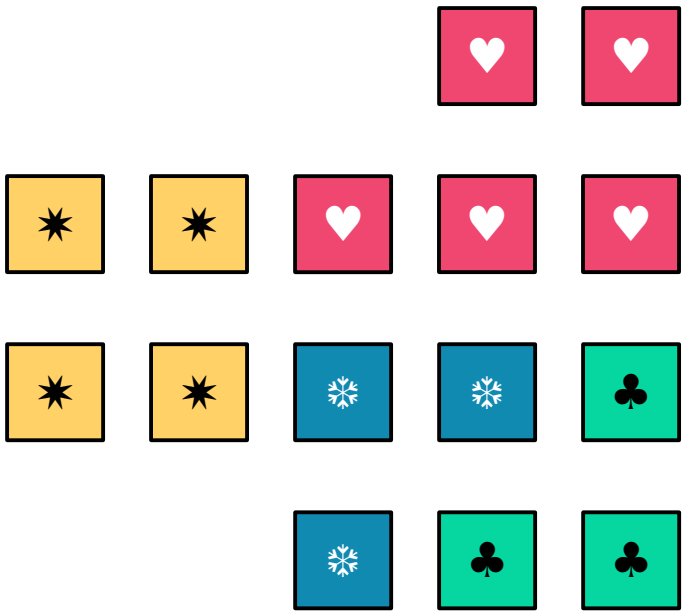

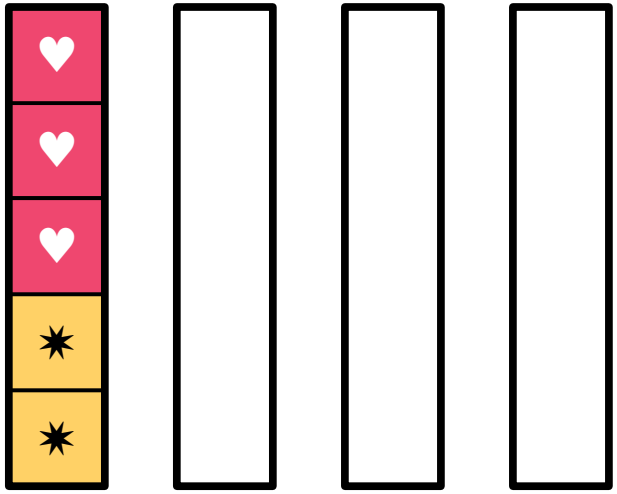

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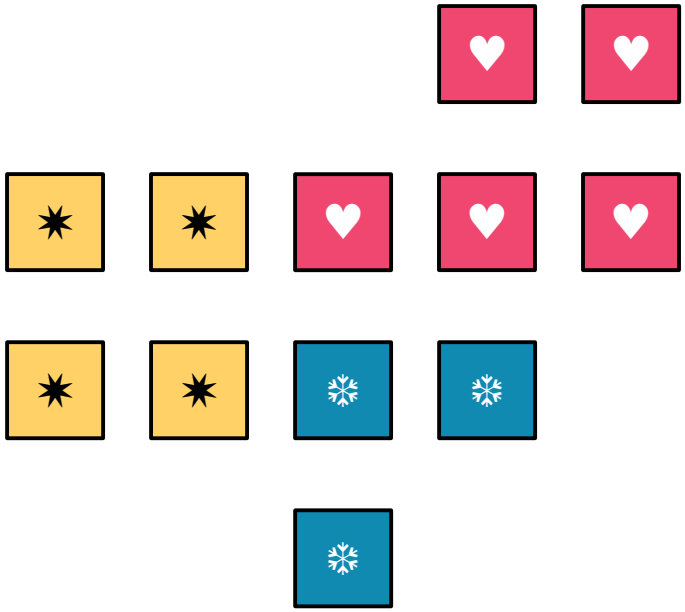

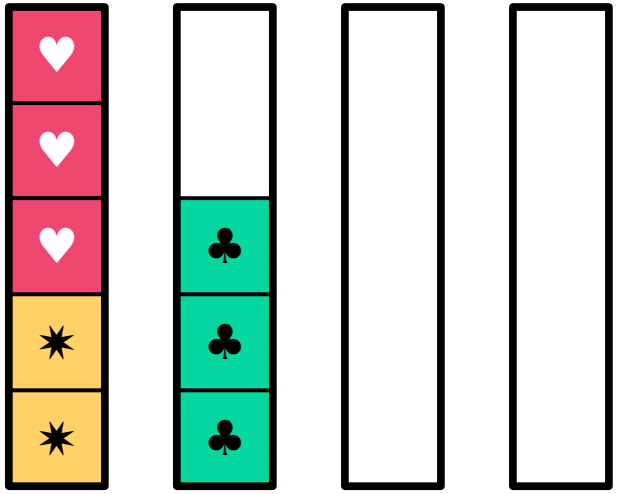

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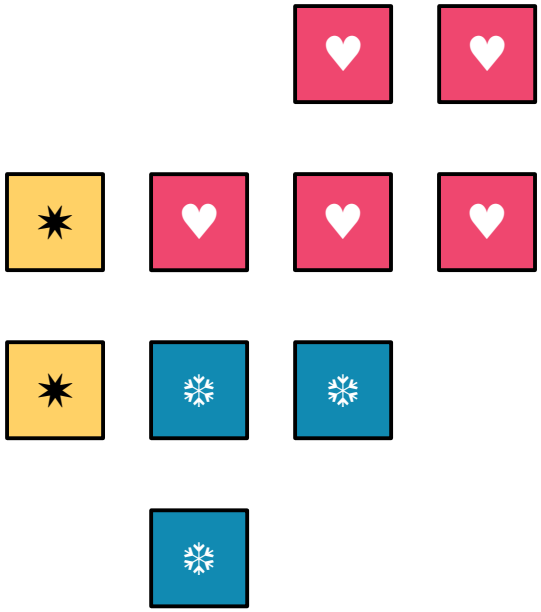

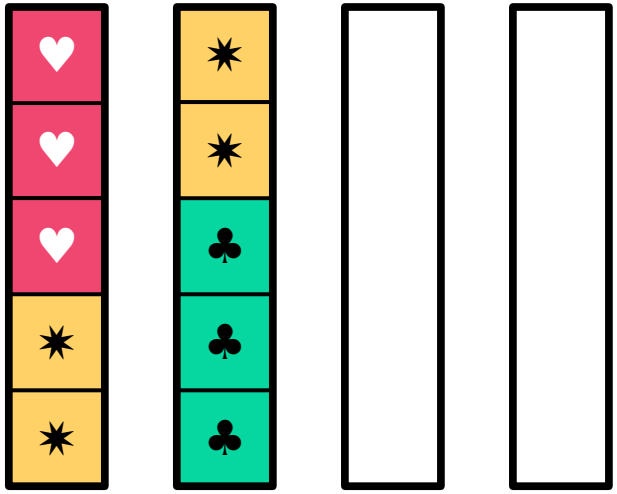

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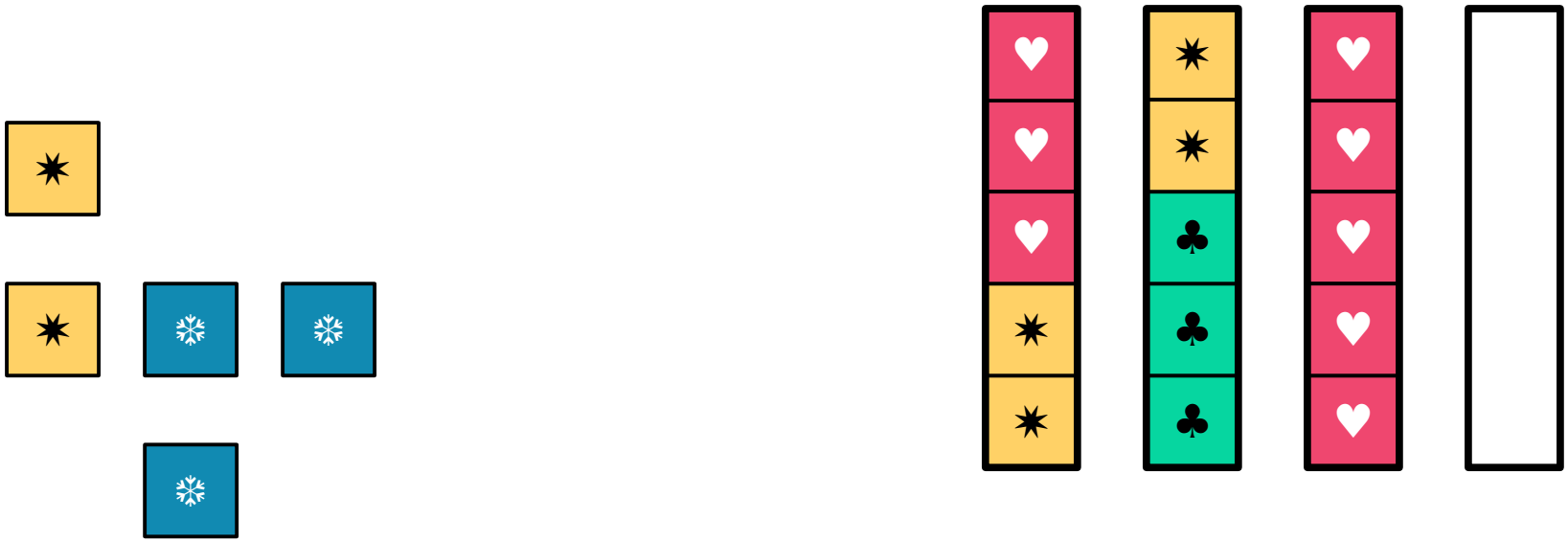

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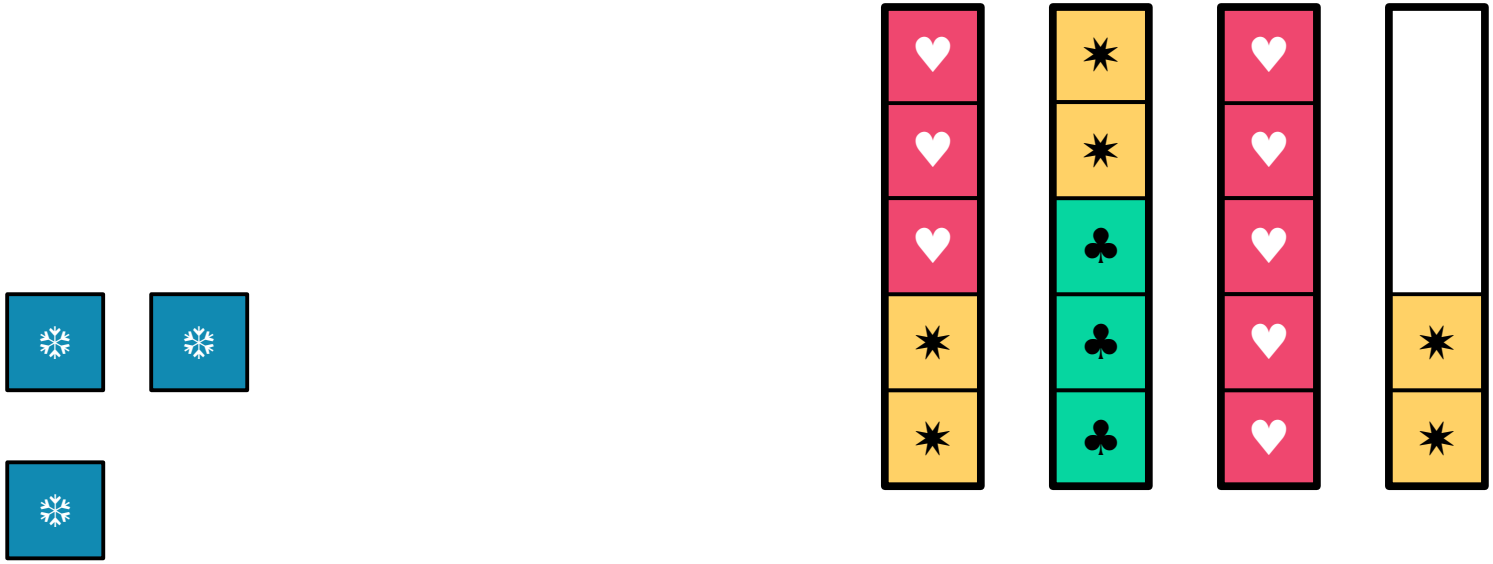

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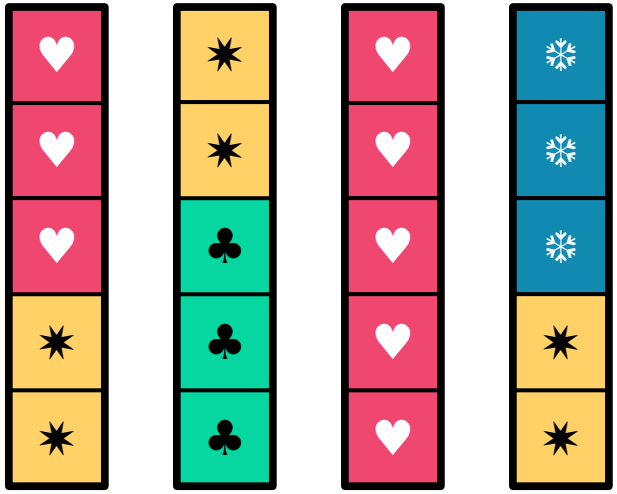

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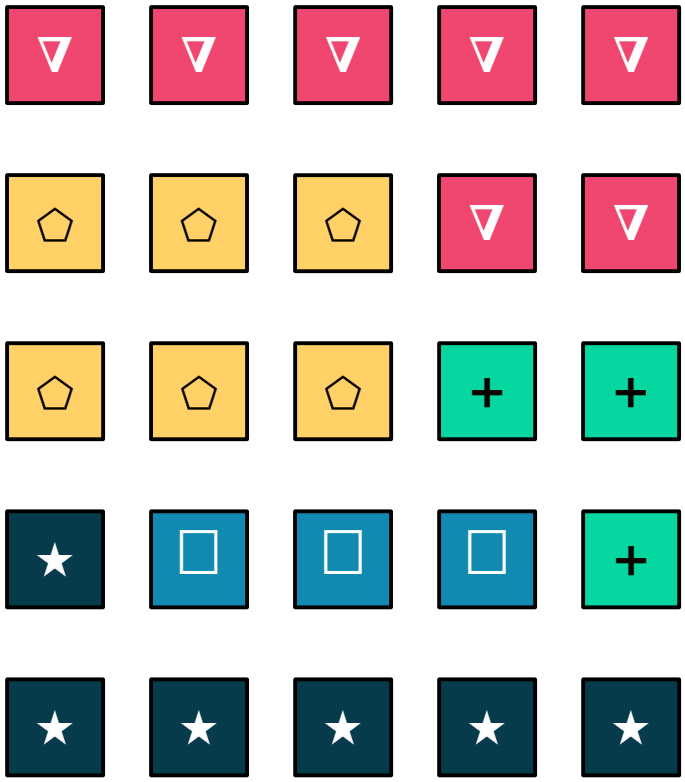
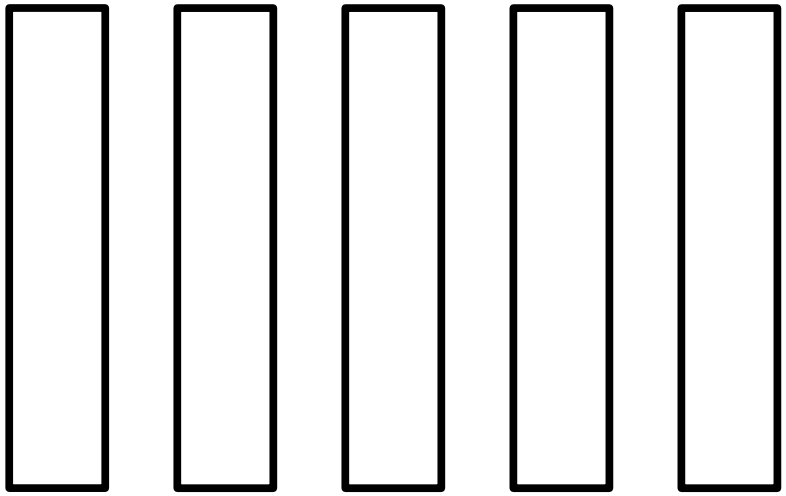
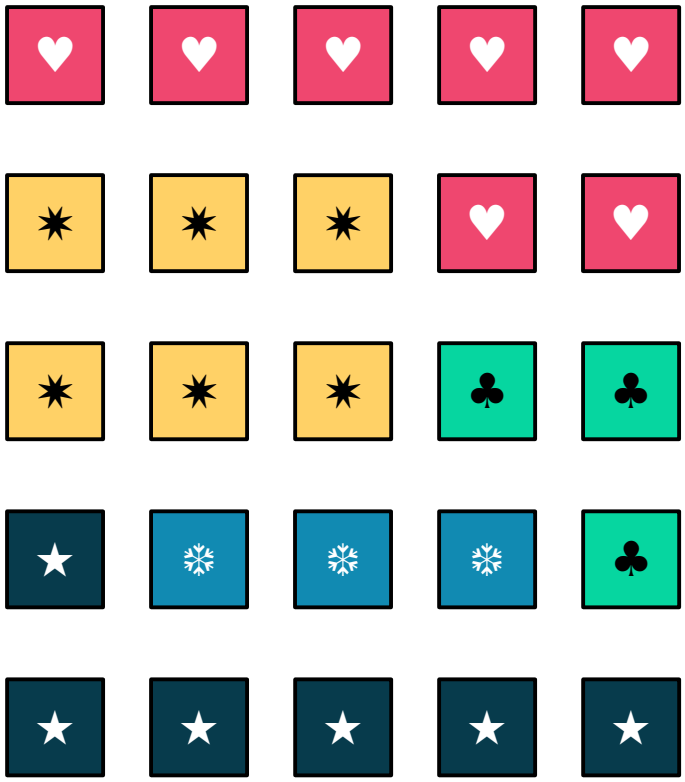
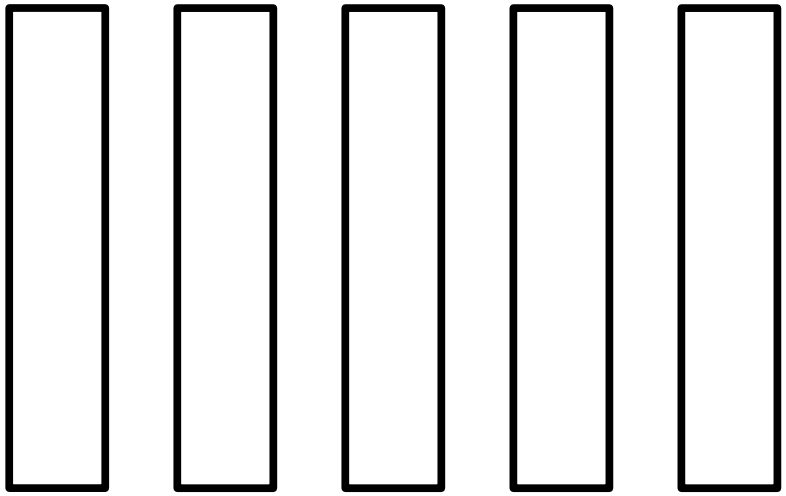




Here are 25 cubes of 5 different colors.
Split them into 5 groups of 5 cubes each so that each group has cubes of at most two different colors.





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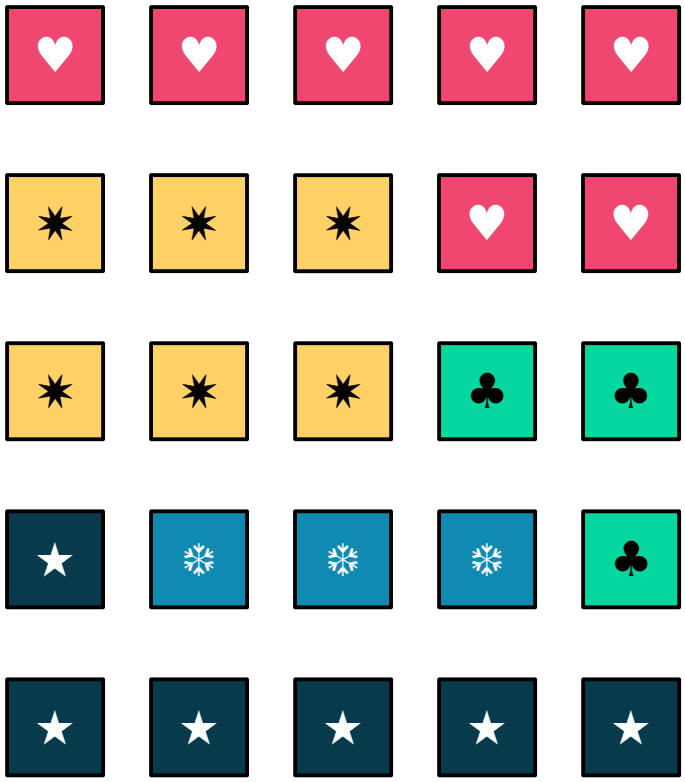

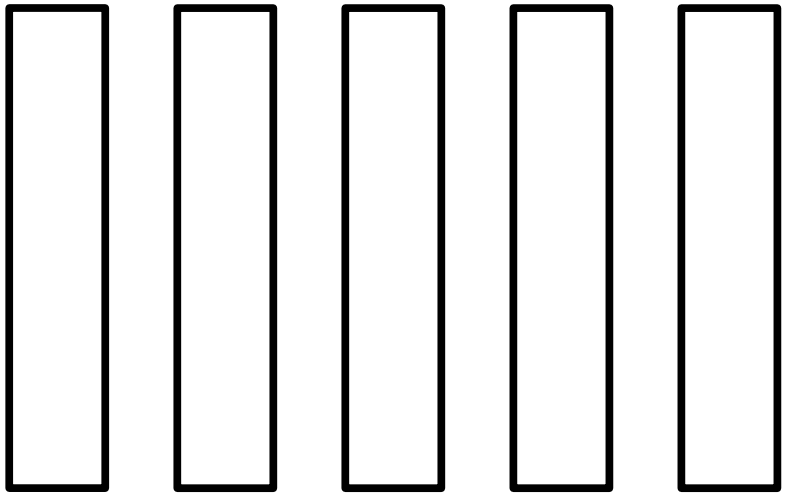

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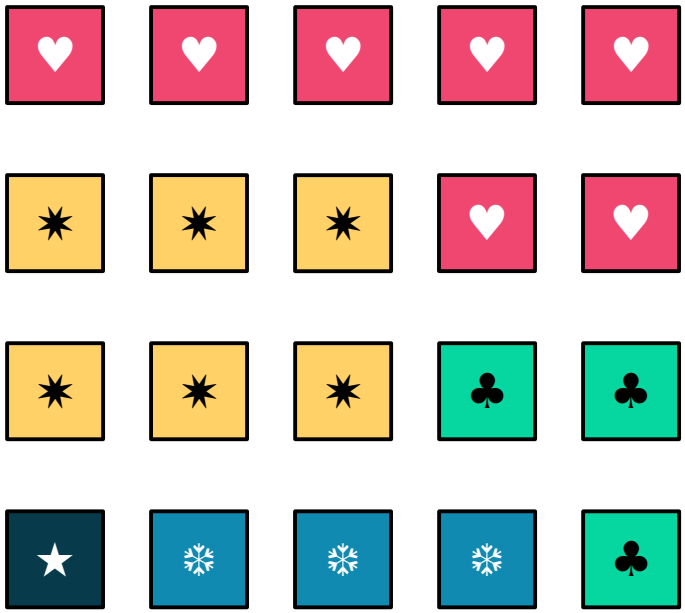

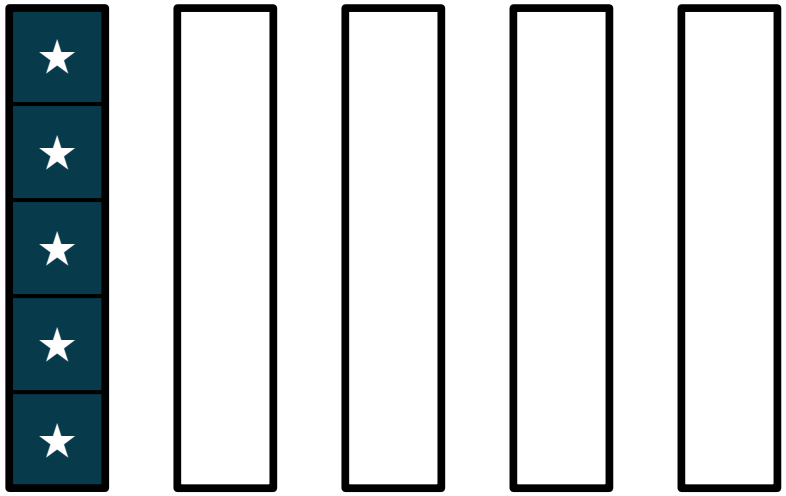

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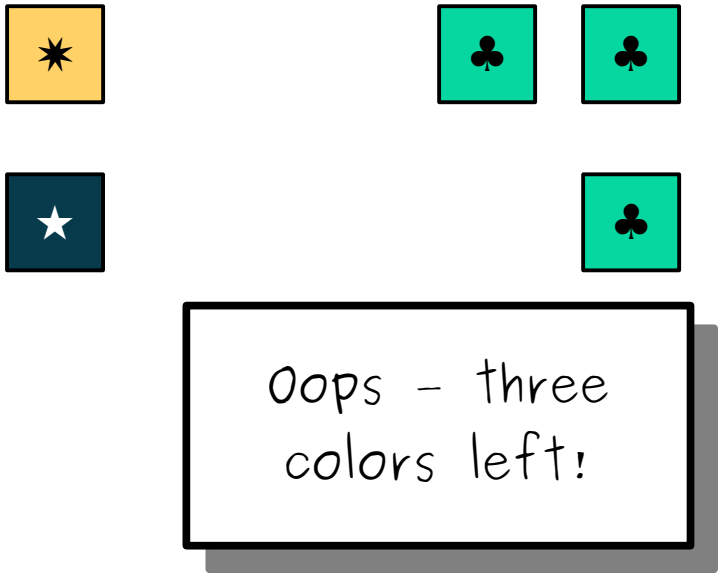

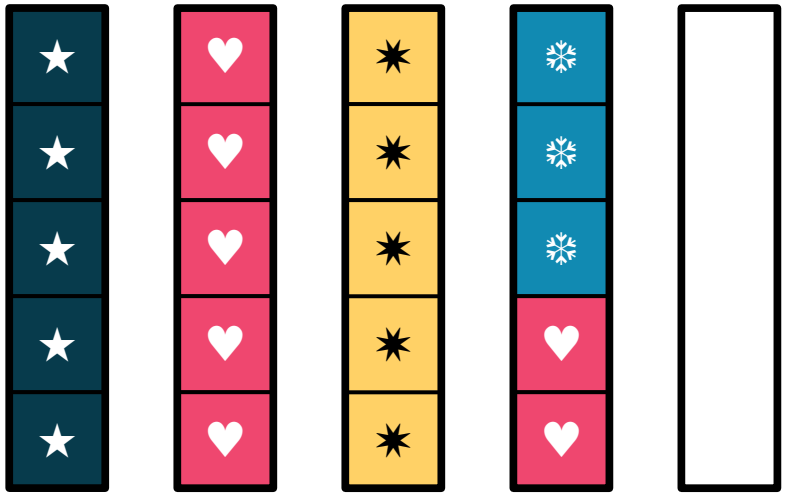

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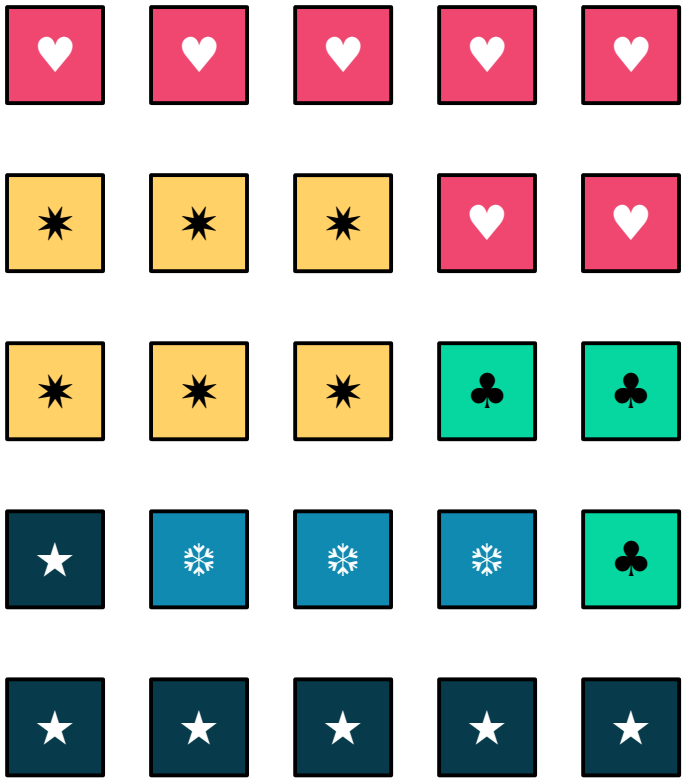

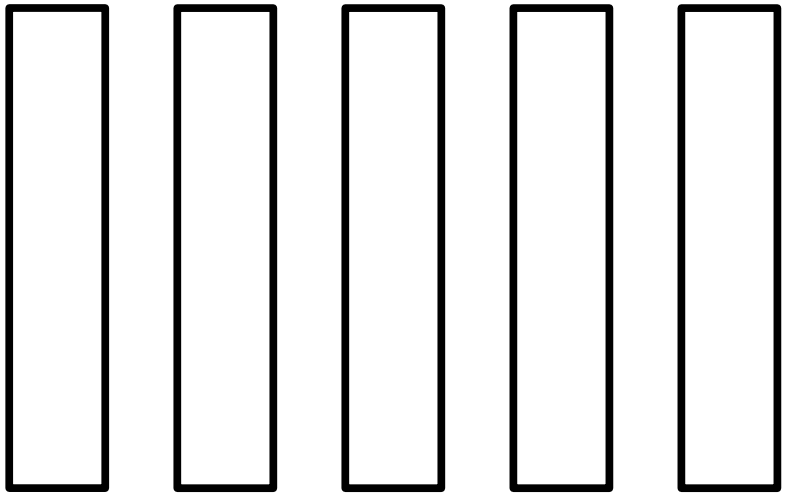

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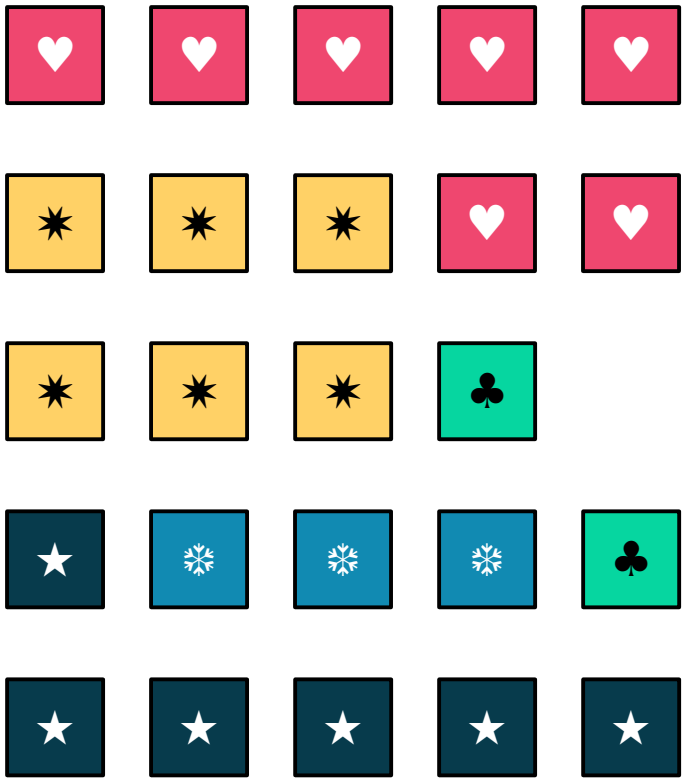

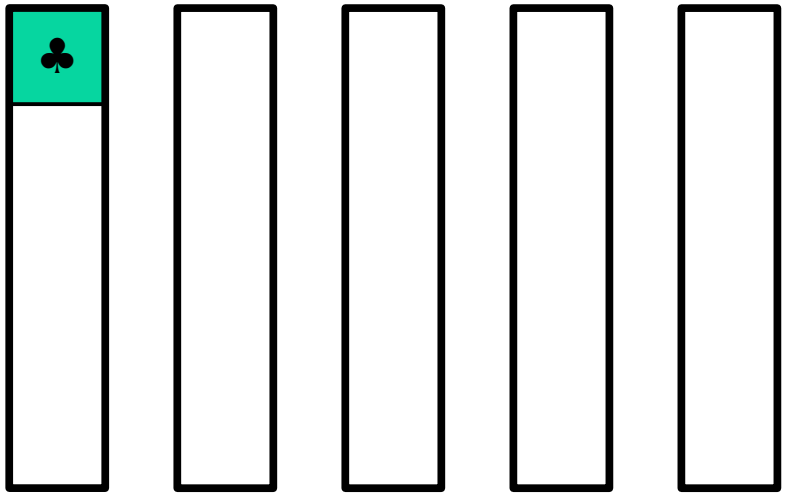

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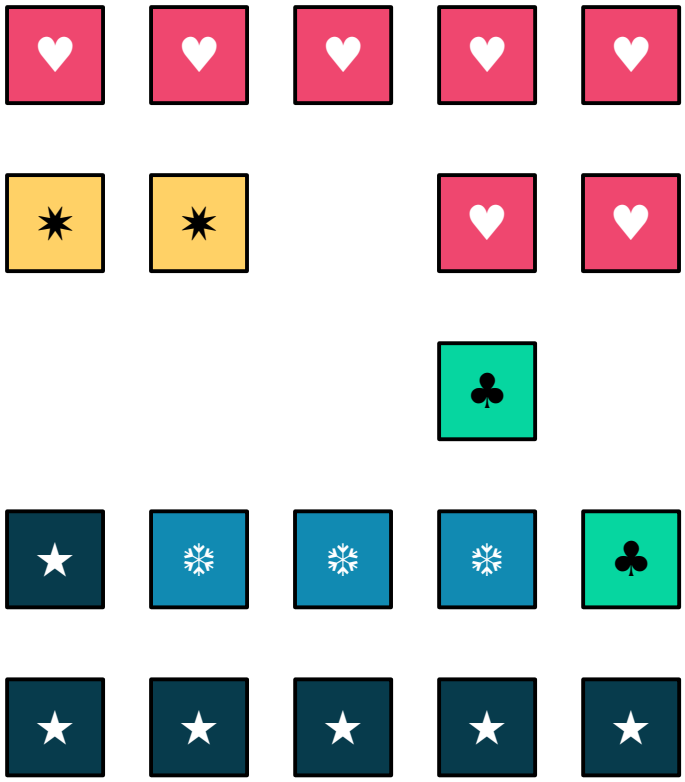

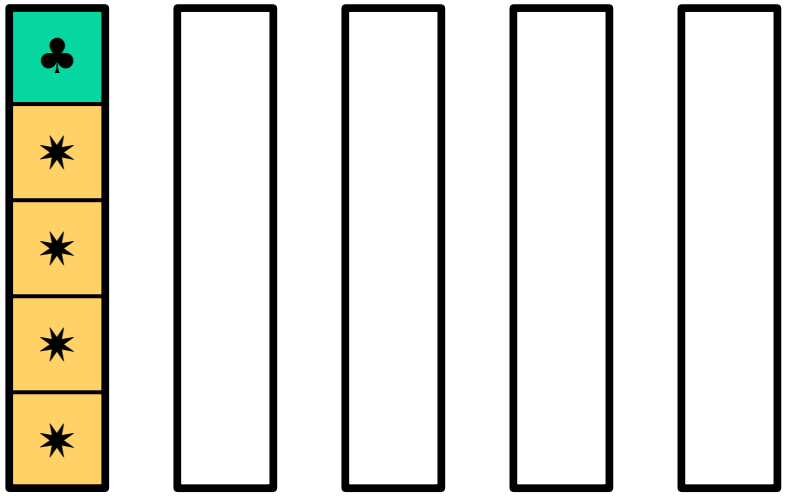

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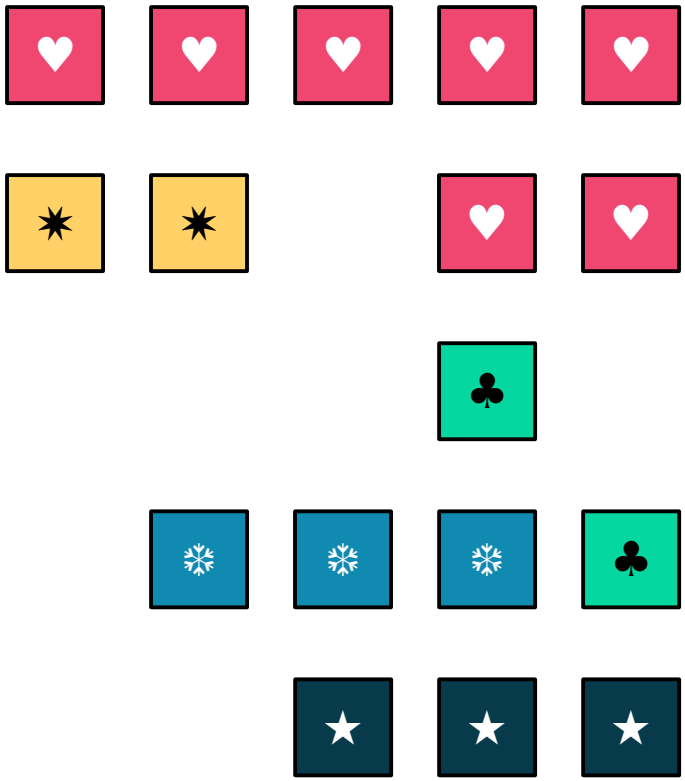

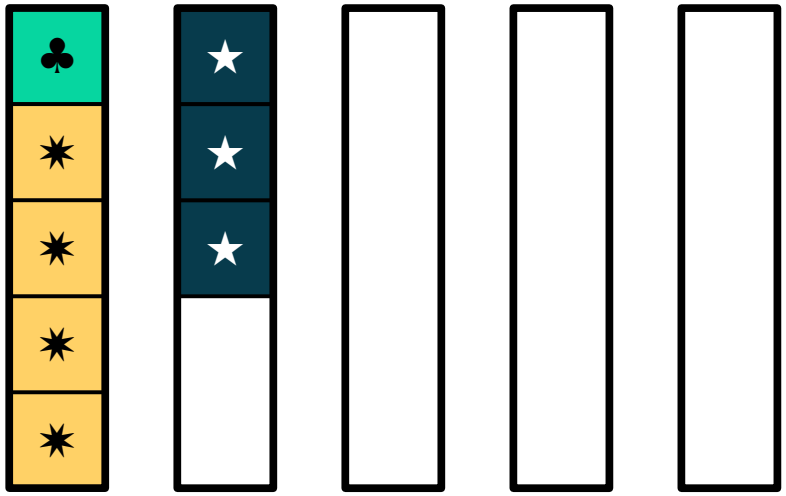

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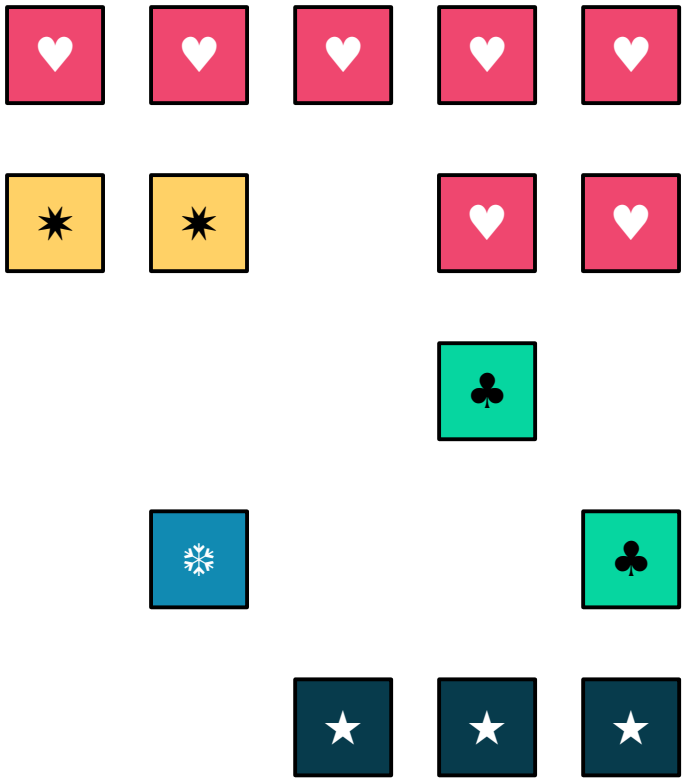

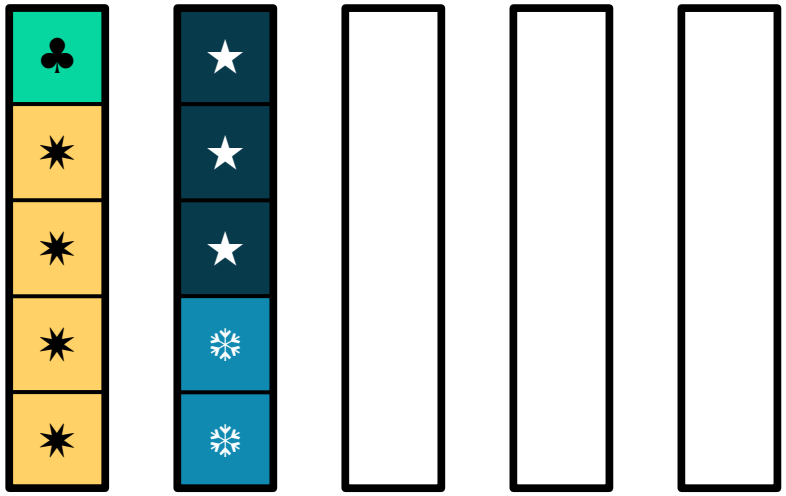

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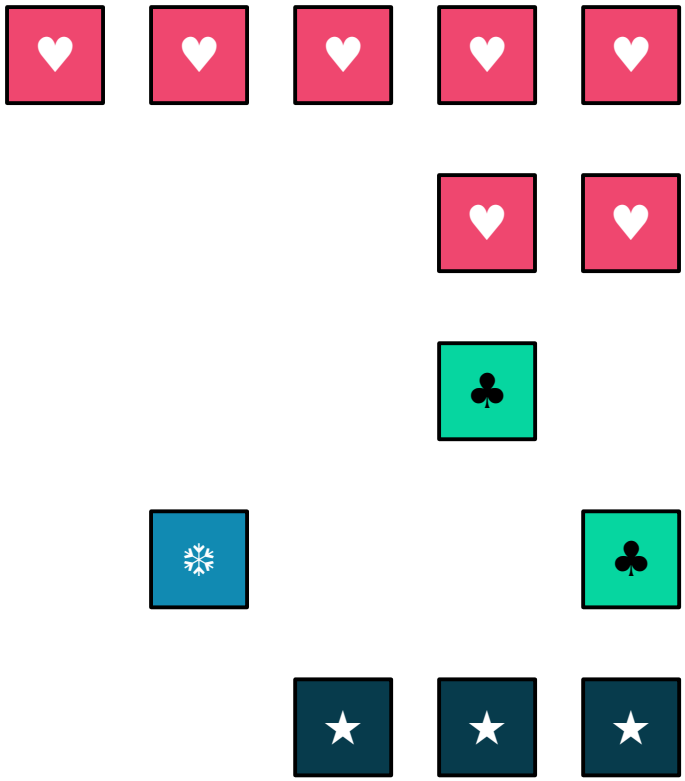

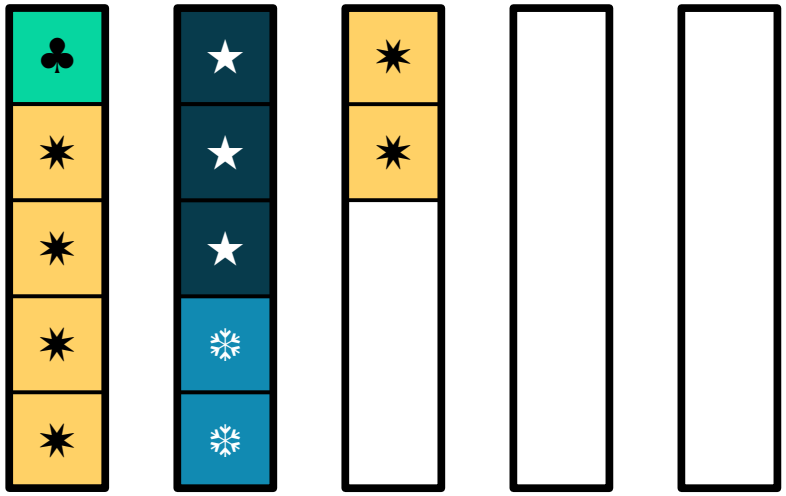

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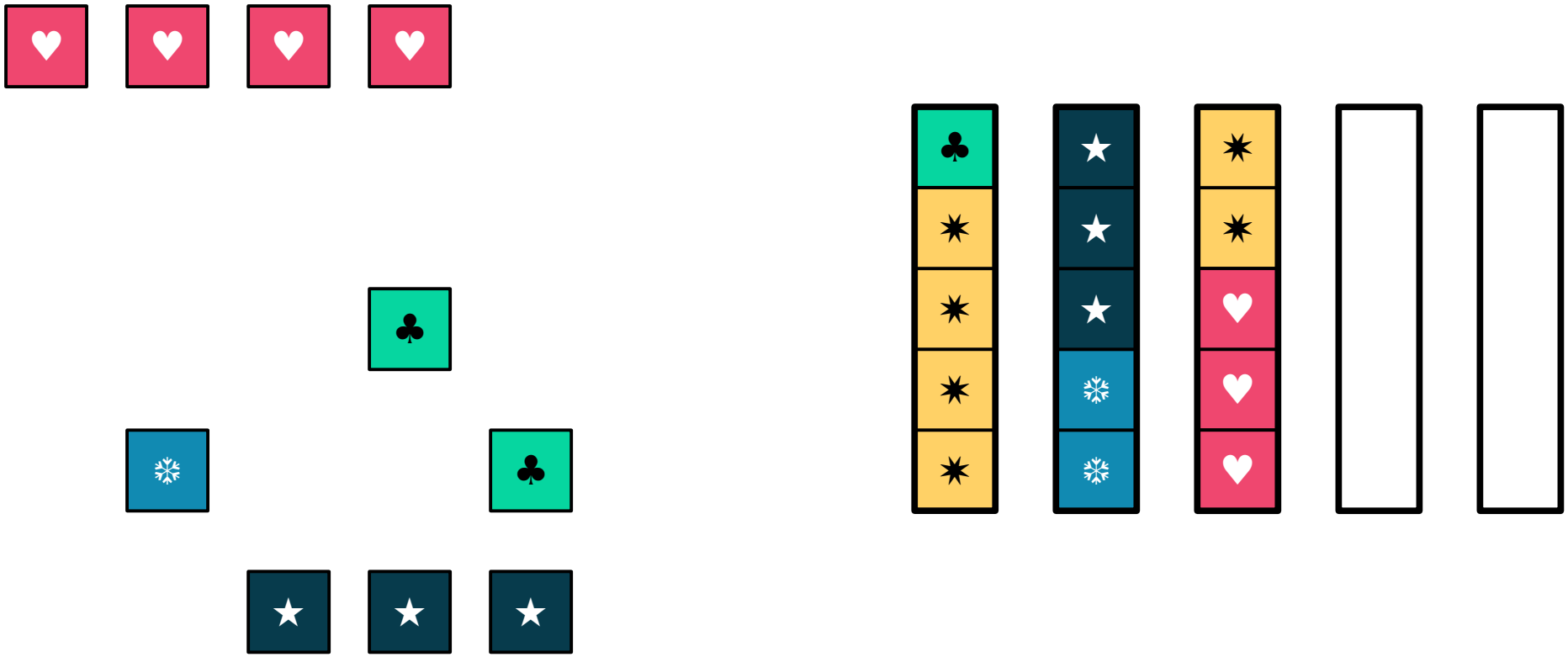

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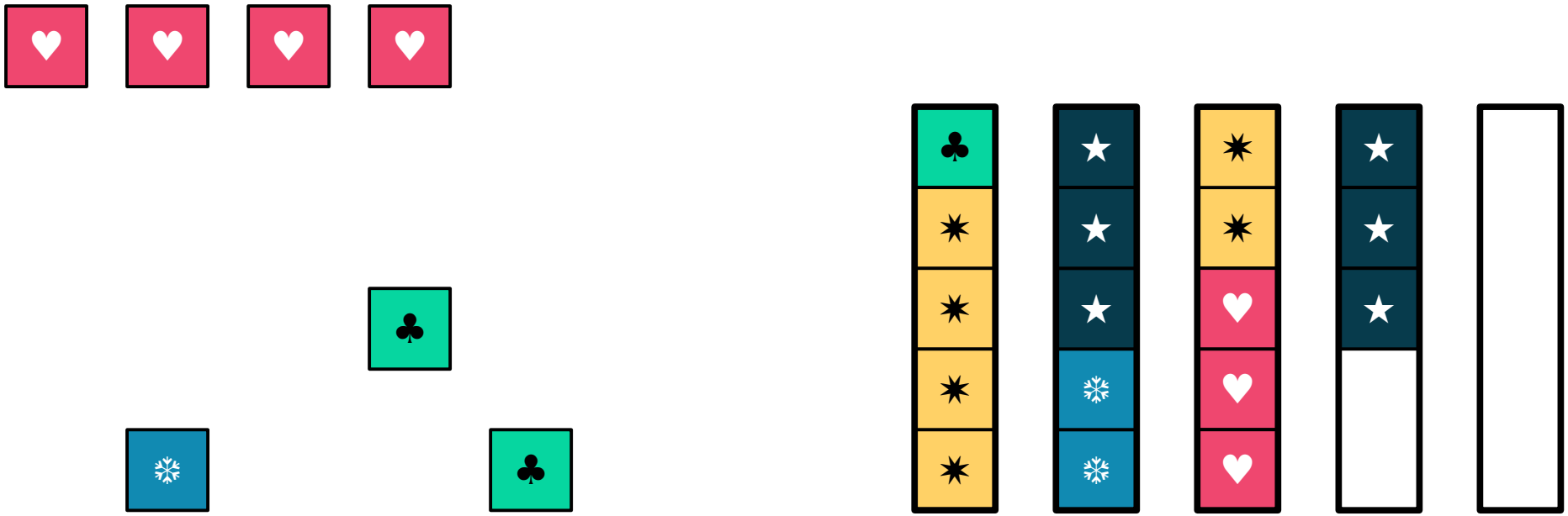

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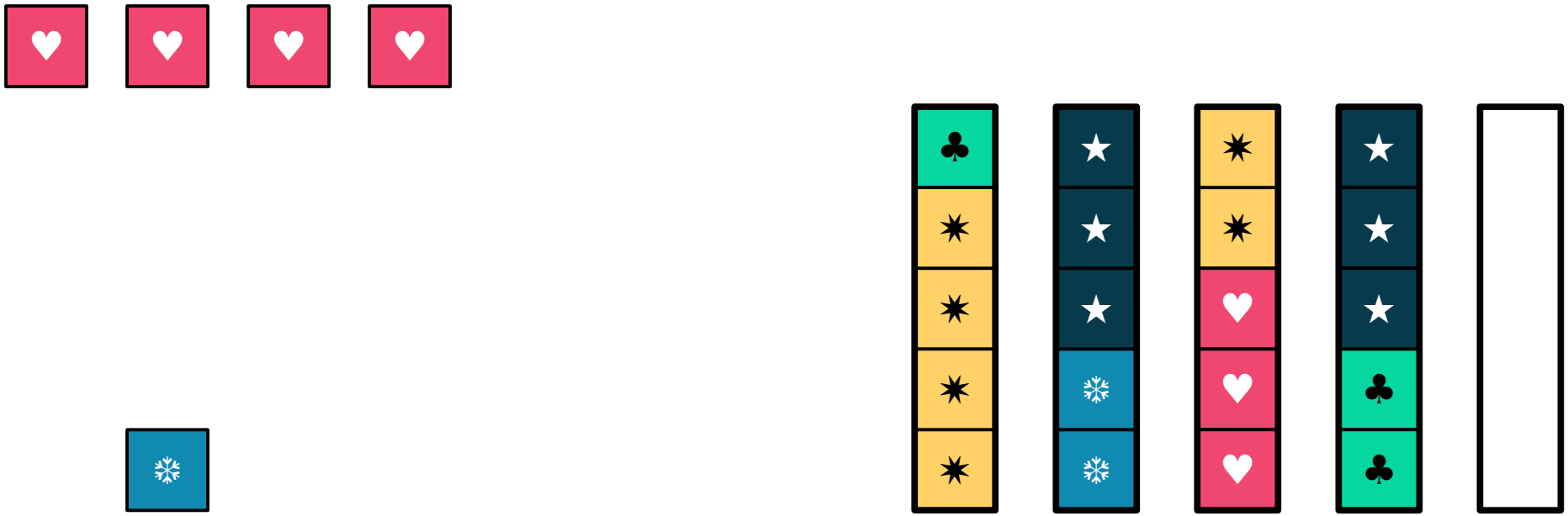

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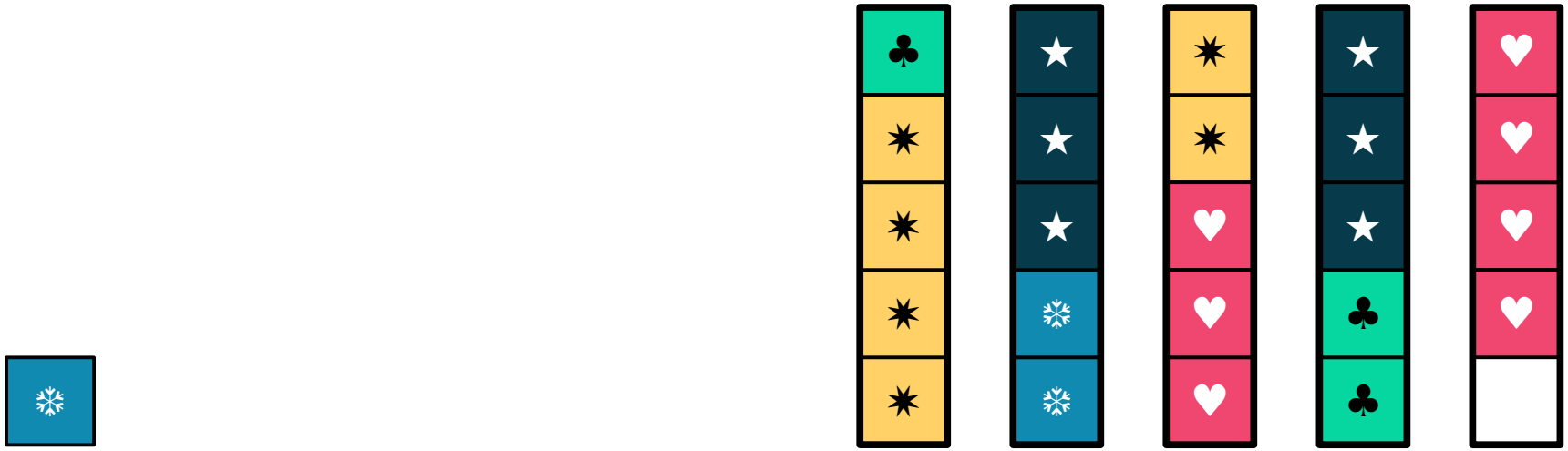

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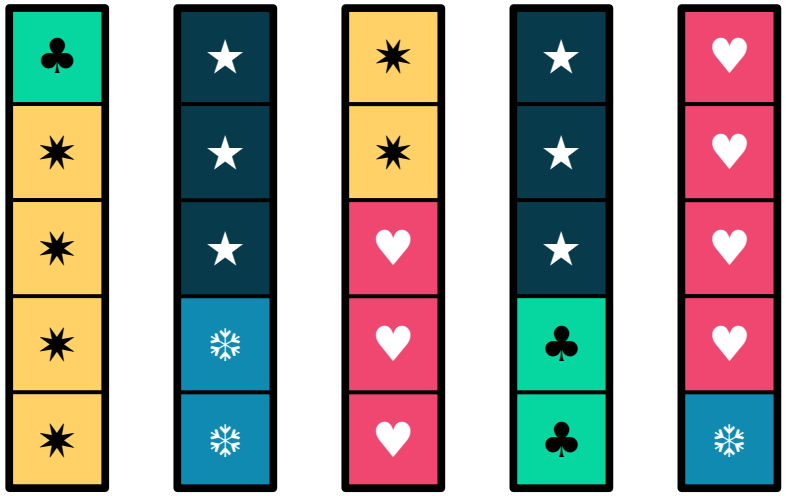

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A ***good split*** of a group of $5n$ potions of n colors is a way of splitting them across shelves where each shelf has five potions with no more than two colors.

Theorem: For any group of $5n$ potions of n colors, there is a good split of those potions.

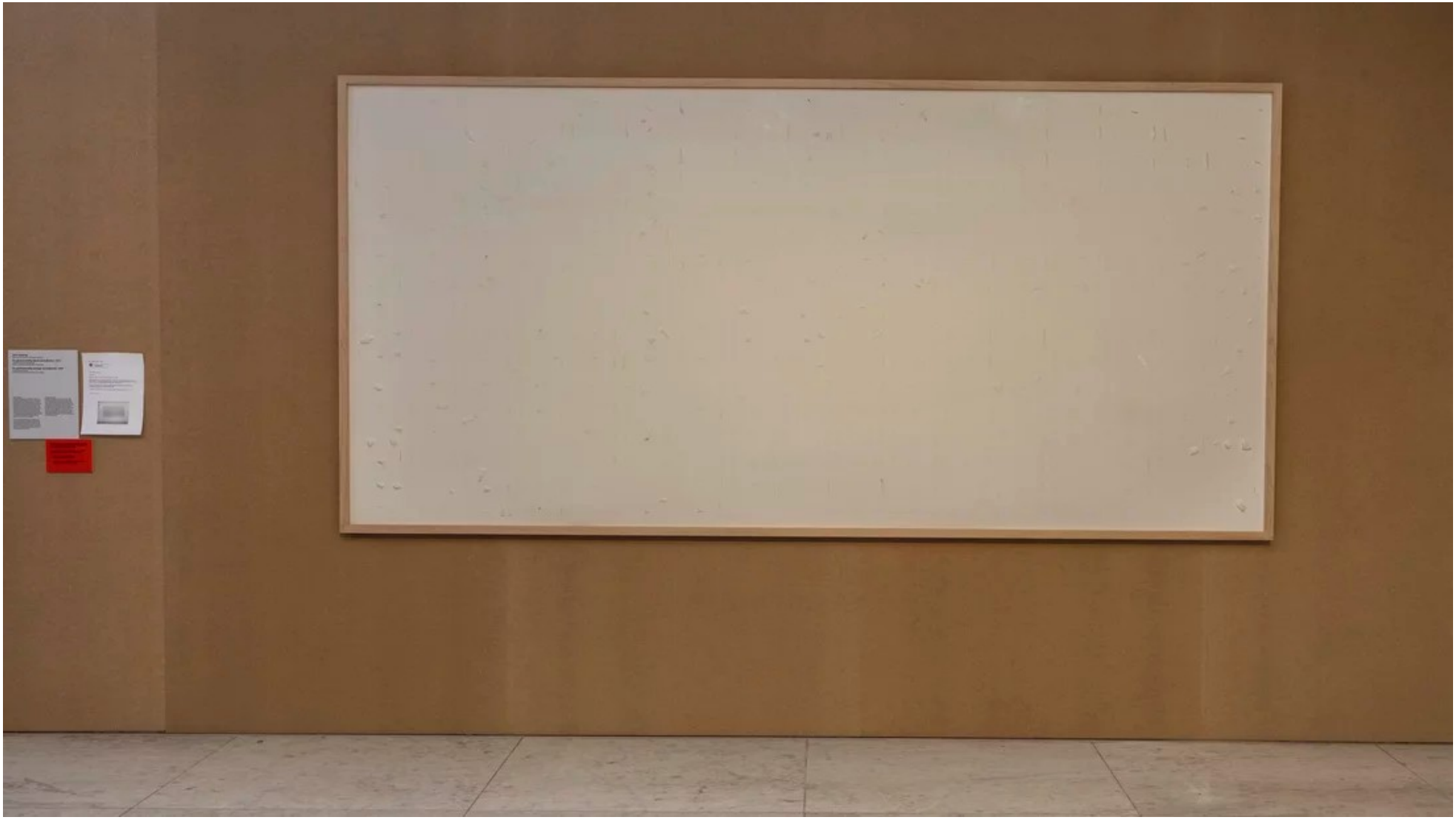
$P(n)$ is the statement "for any group of $5n$ potions of n colors, there exists a good split of those potions."

Basically, do nothing! ➜ $P(0)$

… kind of like
Jens Haaning!

Theorem: For any group of $5n$ potions of n different, colors there exists a good split of those potions.

“Take the Money and Run”



Jens Haaning, 2021

P(n) is the statement "for any group of $5n$ potions of n colors, there exists a good split of those potions."

$$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$$

Which of the following best describes the high-level structure of the inductive step of this proof?

A. Begin with a group of $5k$ potions of k colors. Find a way to add in five new potions and one color.

B. Begin with a group of $5k+5$ potions of $k+1$ colors. Find a way to remove five potions and one color.

Answer at ***https://cs103.stanford.edu/pollev***

Theorem: For any group of $5n$ potions of n different, colors there exists a good split of those potions.

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What kind of quantifier is embedded in our predicate, P(n)?

Theorem: For any group of $5n$ potions of n different, colors there exists a good split of those potions.

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if

for every group of $5k$ potions of k colors, there's a good split of those potions.

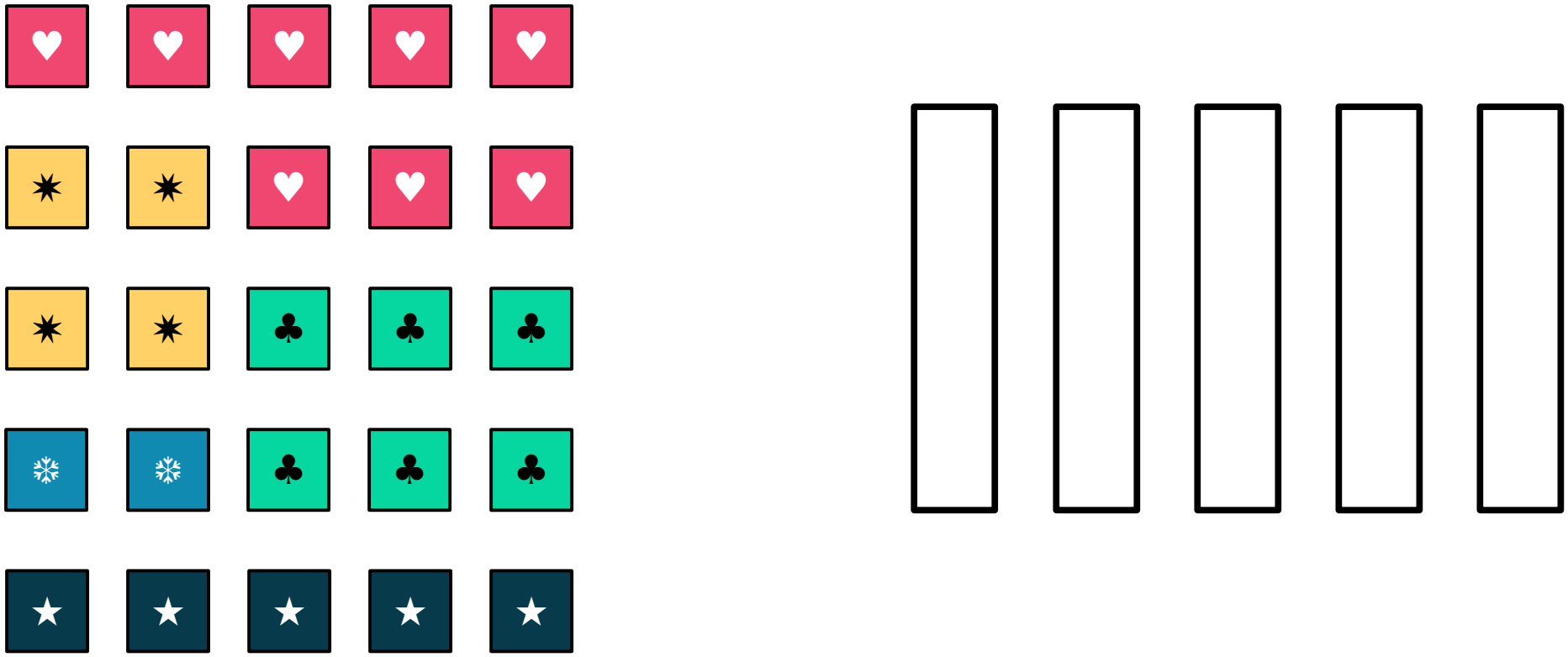
Assume a universal

then

for every group of $5k+5$ potions of $k+1$ colors, there's a good split of those potions.

Prove a universal

Theorem: For any group of $5n$ potions of n different, colors there exists a good split of those potions.



Idea: Begin with $5k+5$ potions and $k+1$ colors.
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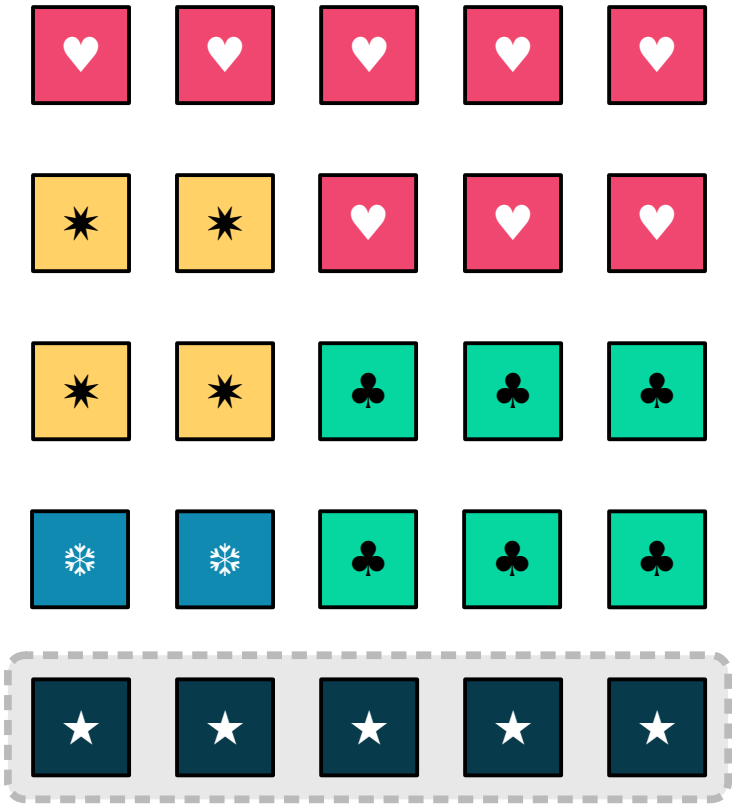

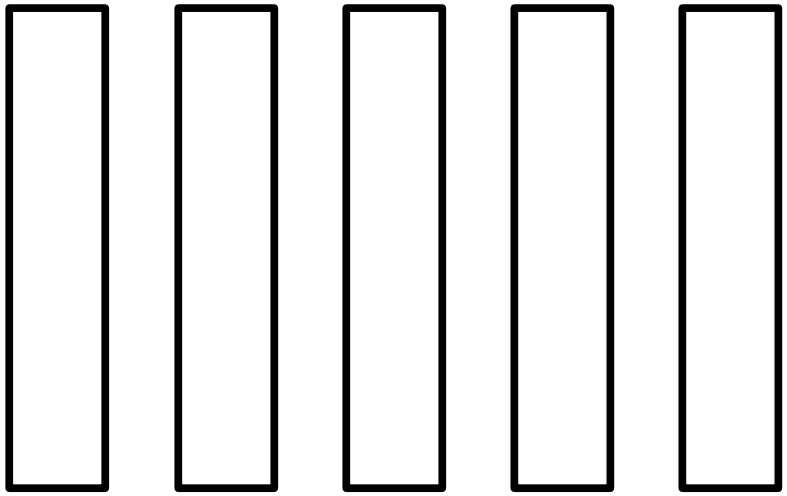

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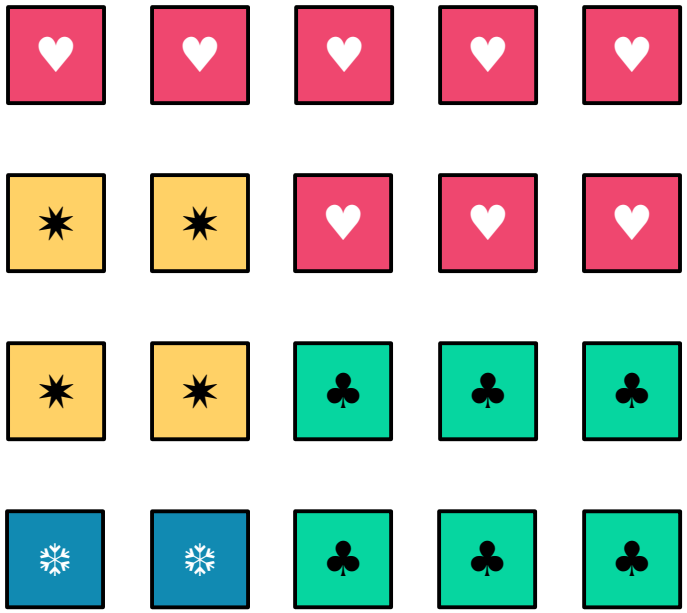

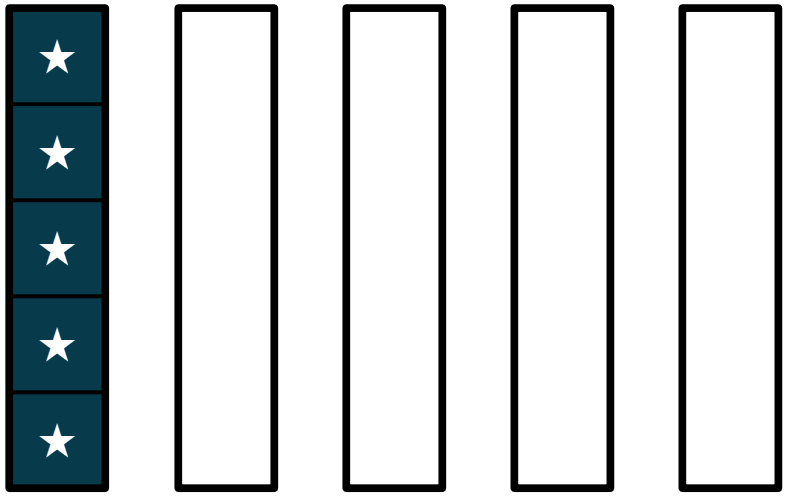

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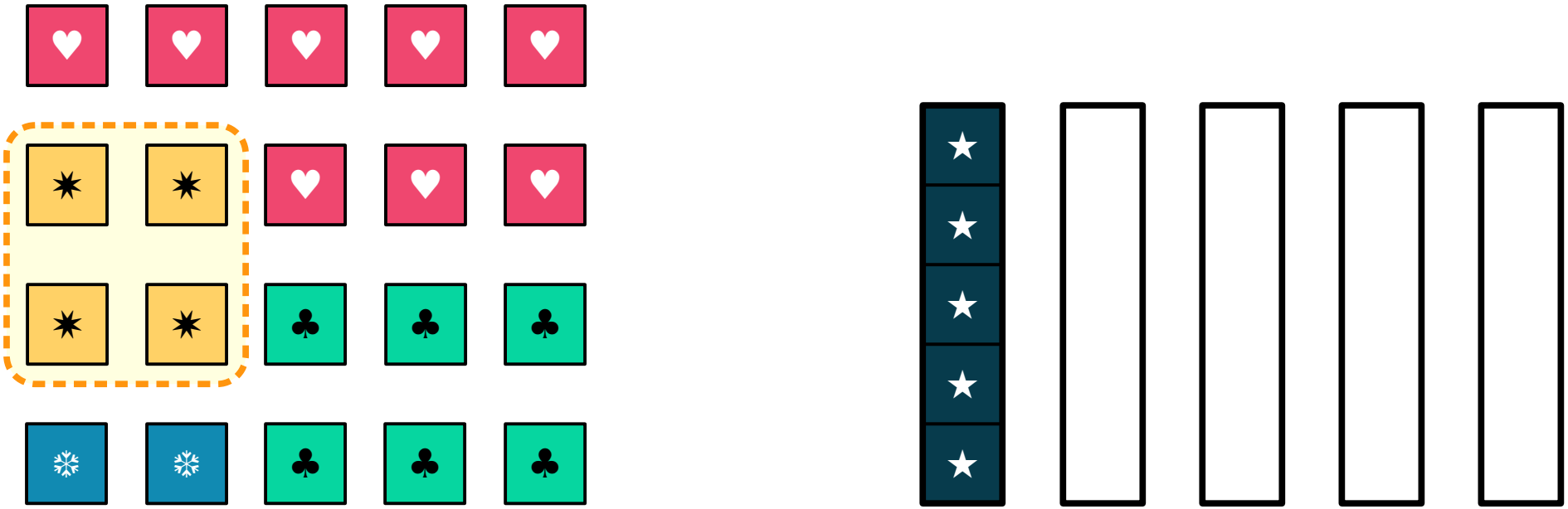

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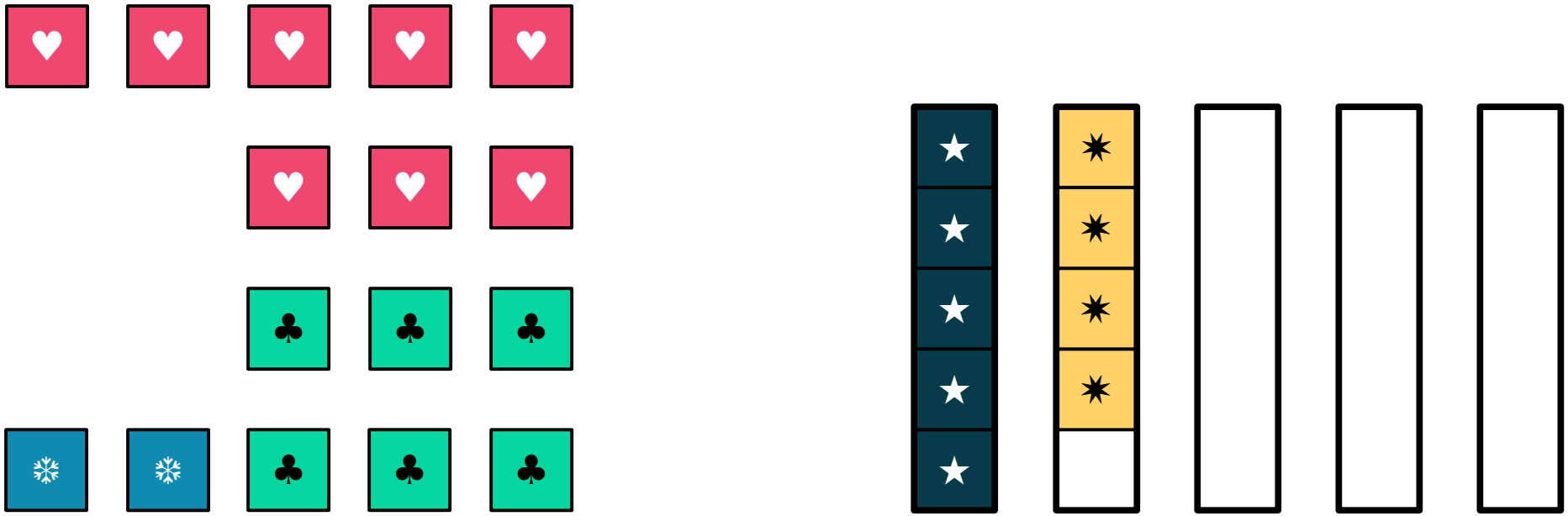

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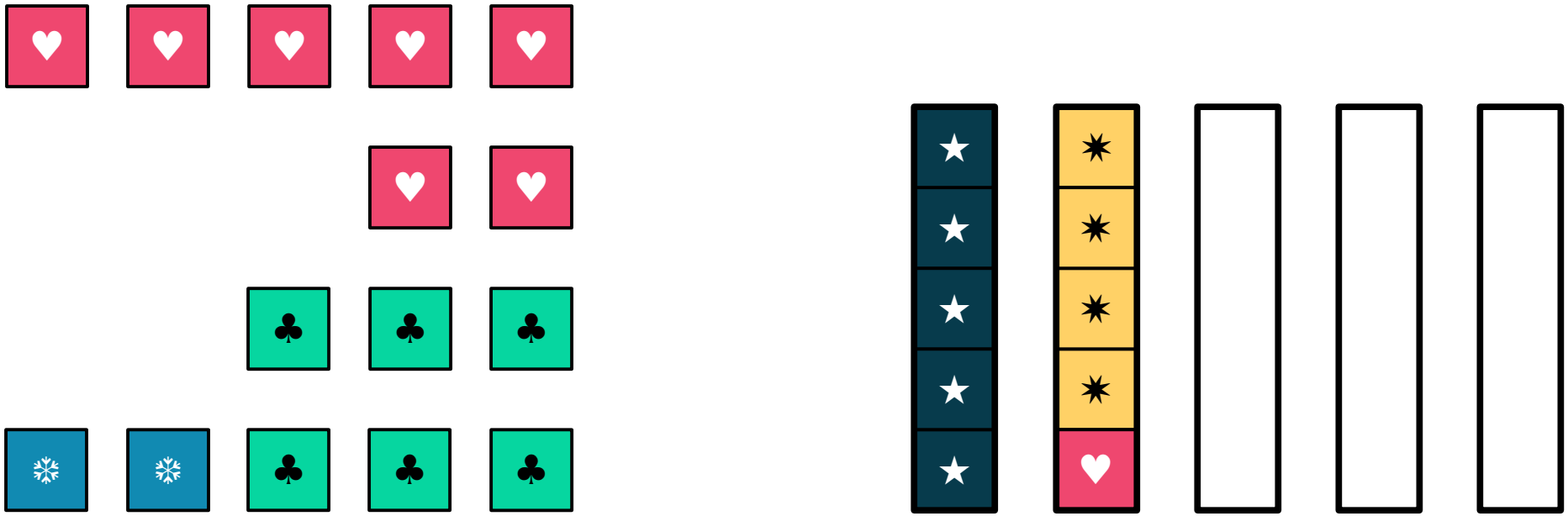

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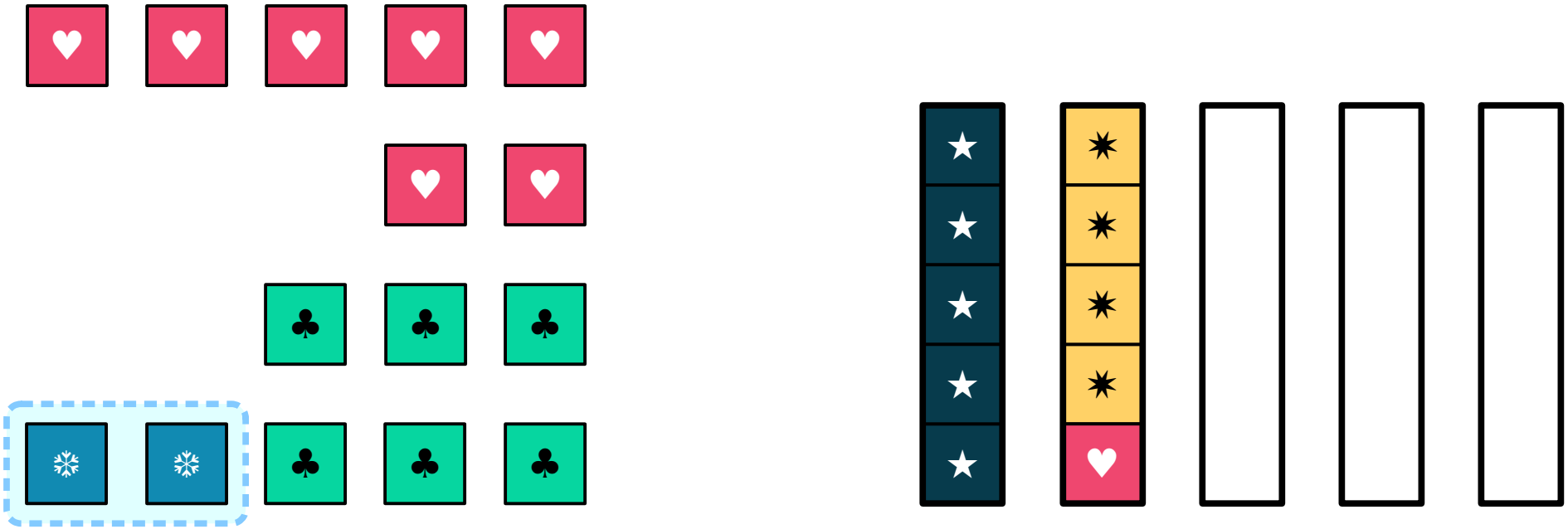

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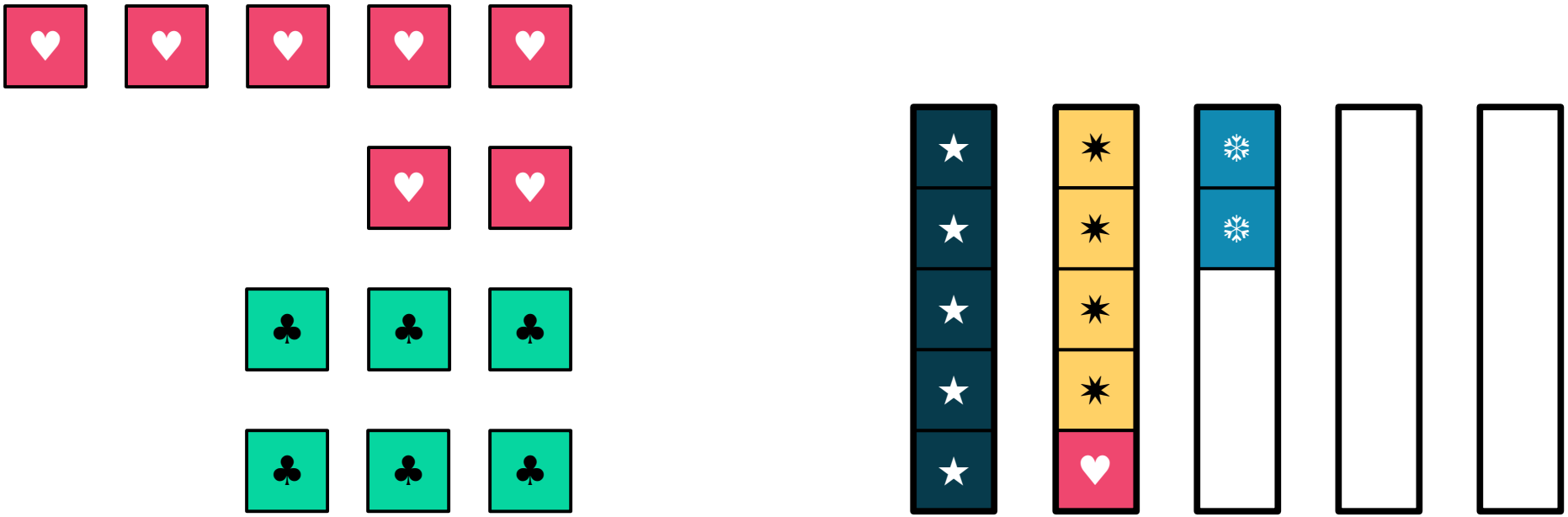

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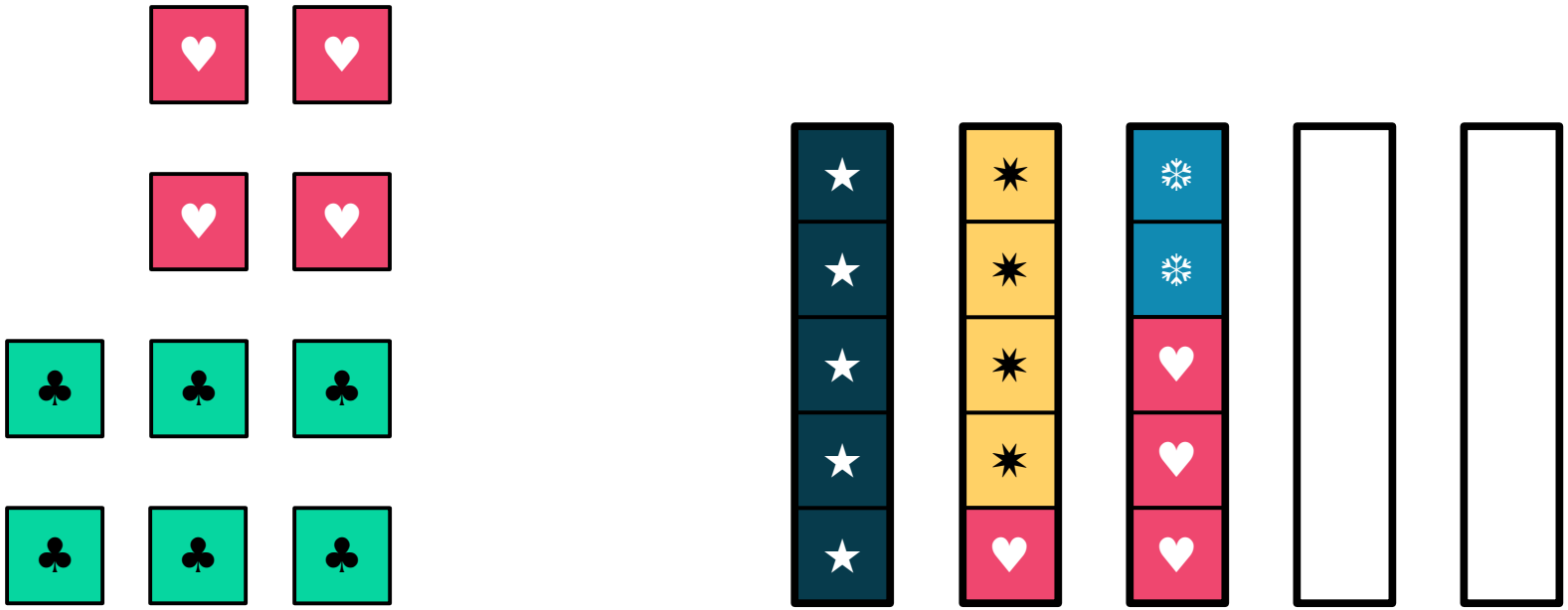

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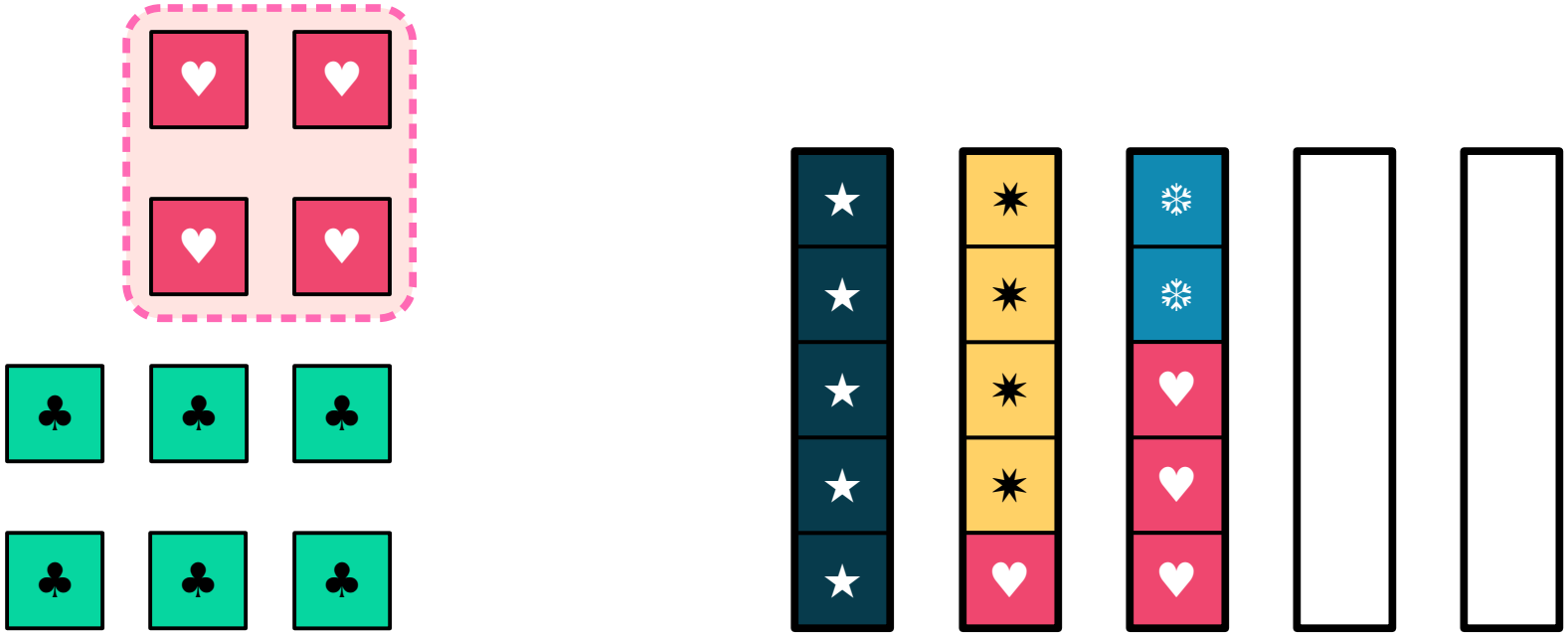

Idea: Begin with $5k+5$ potions and $k+1$ colors.
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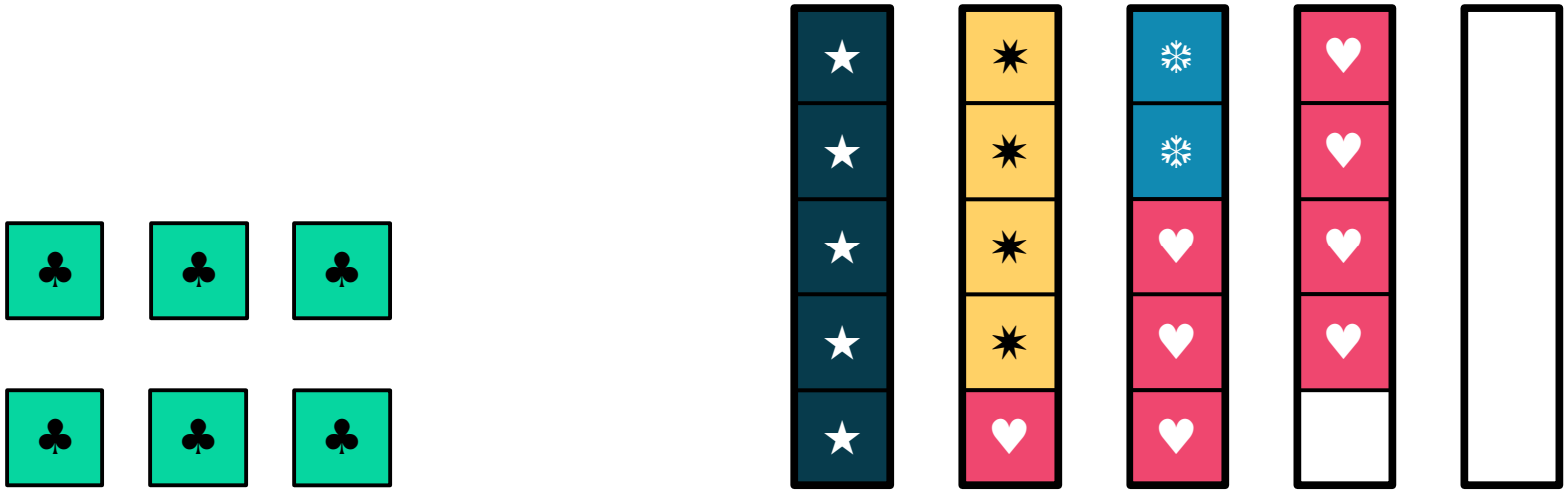

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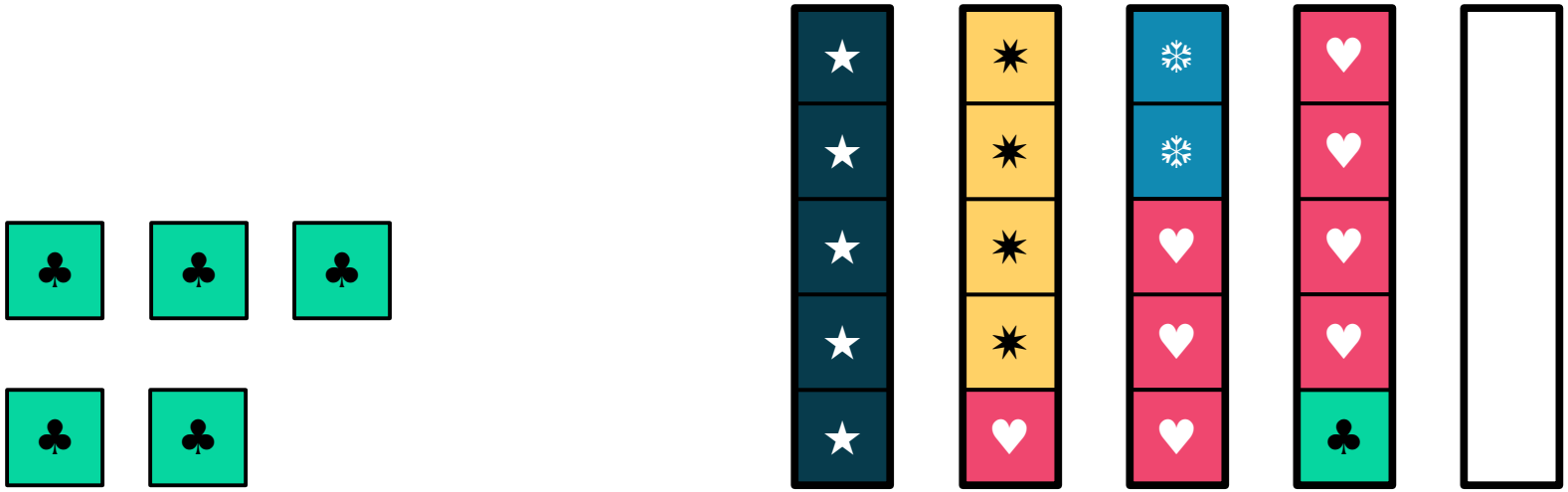

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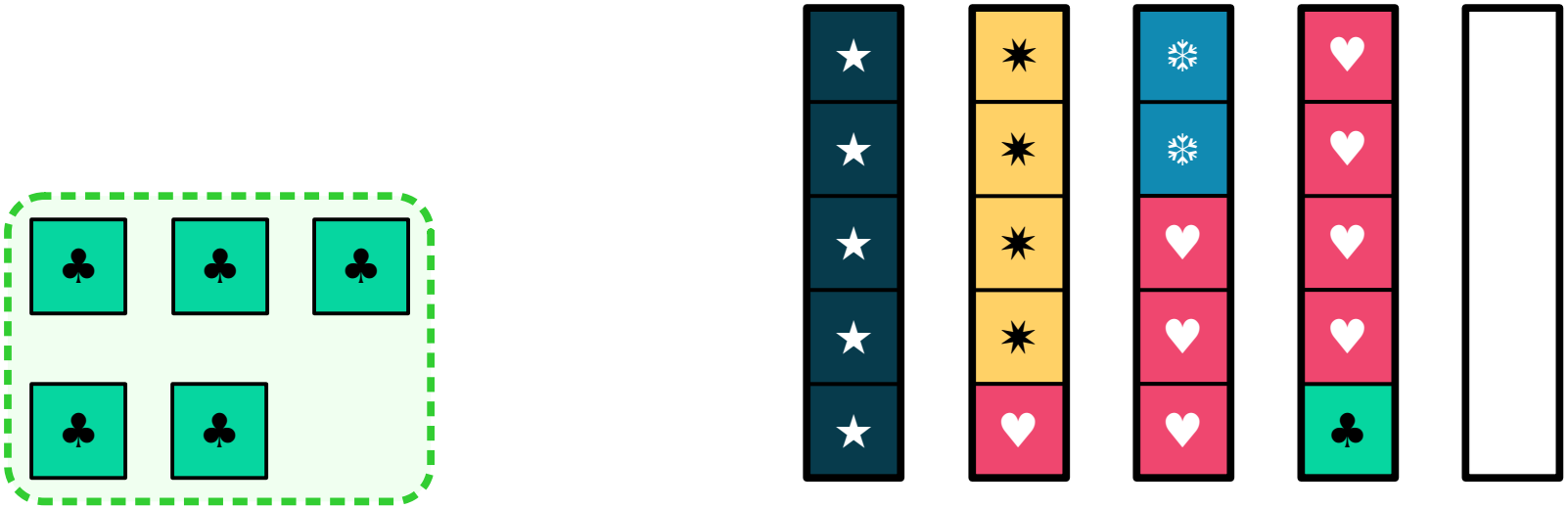

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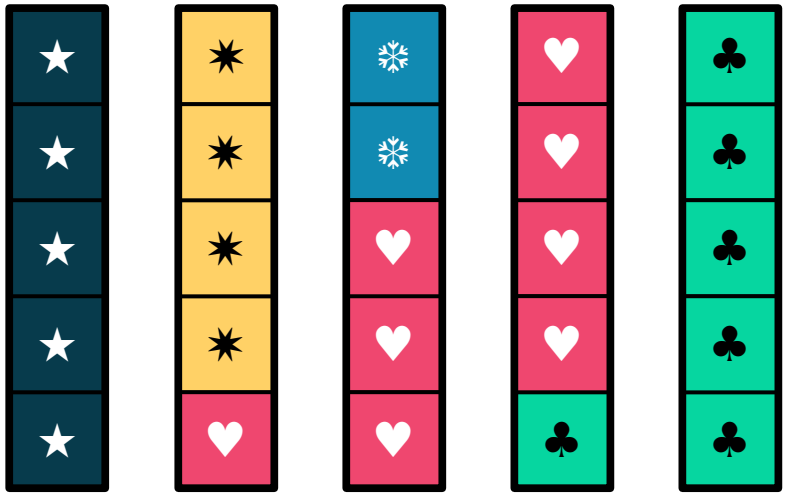

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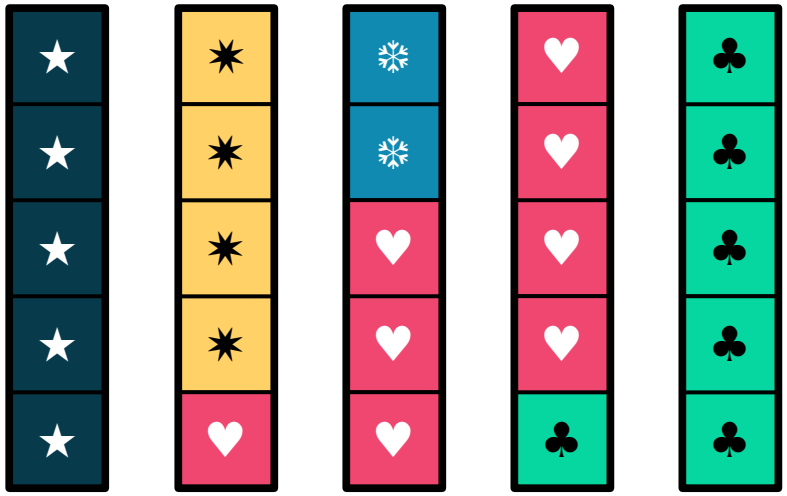
Find a color that appears five or fewer times.

If it's exactly five times, place all potions of that color on a single shelf.

Otherwise, place all potions of that color and "top off" with potions of another color.



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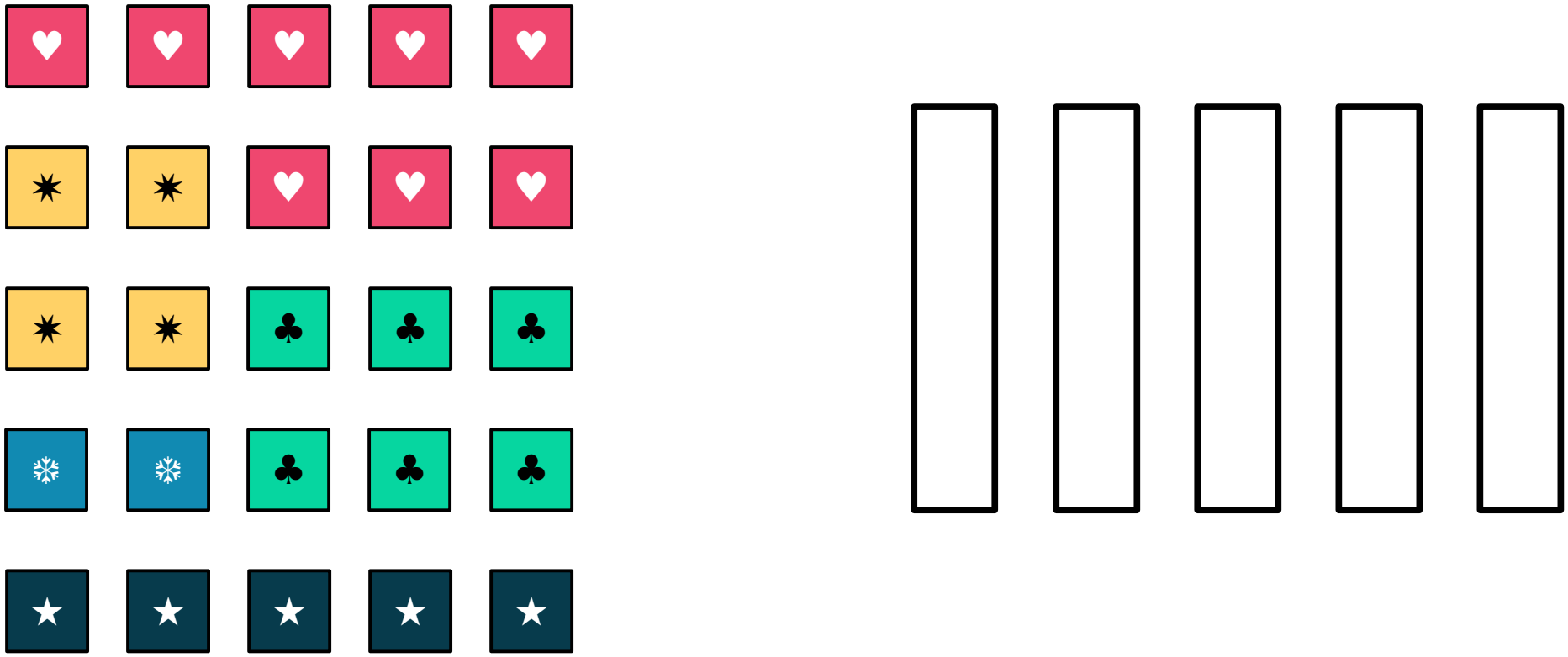

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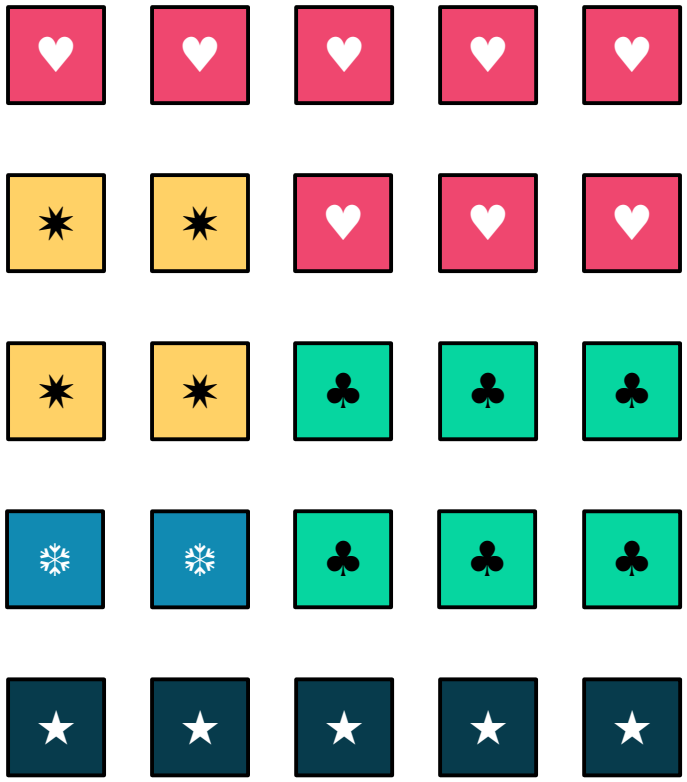

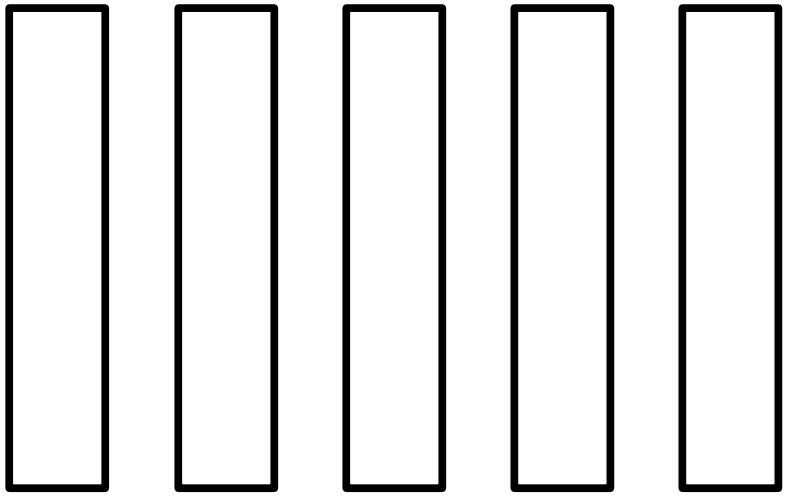

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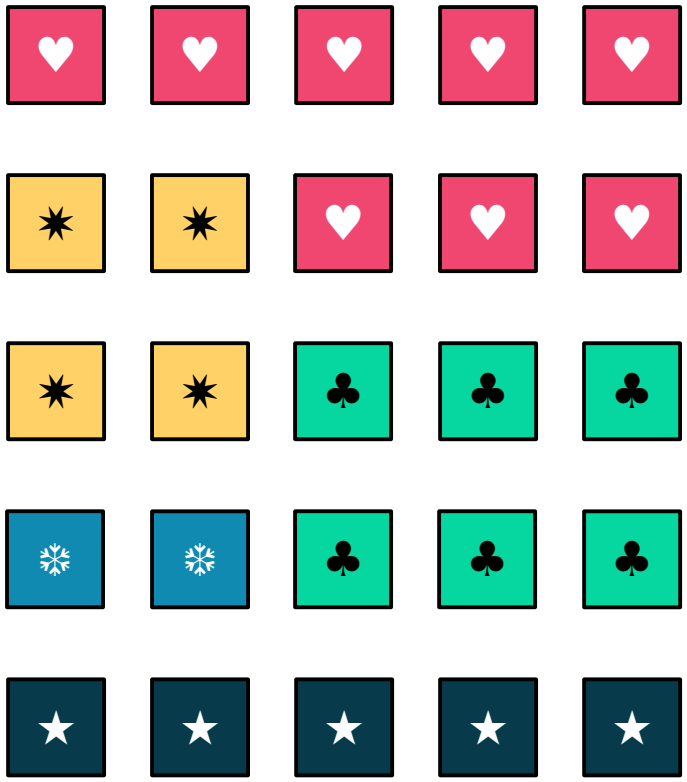

We need to find a color that appears five or fewer times. What mathematical tool guarantees such a color exists?

Answer at:
https://cs103.stanford.edu/pollev

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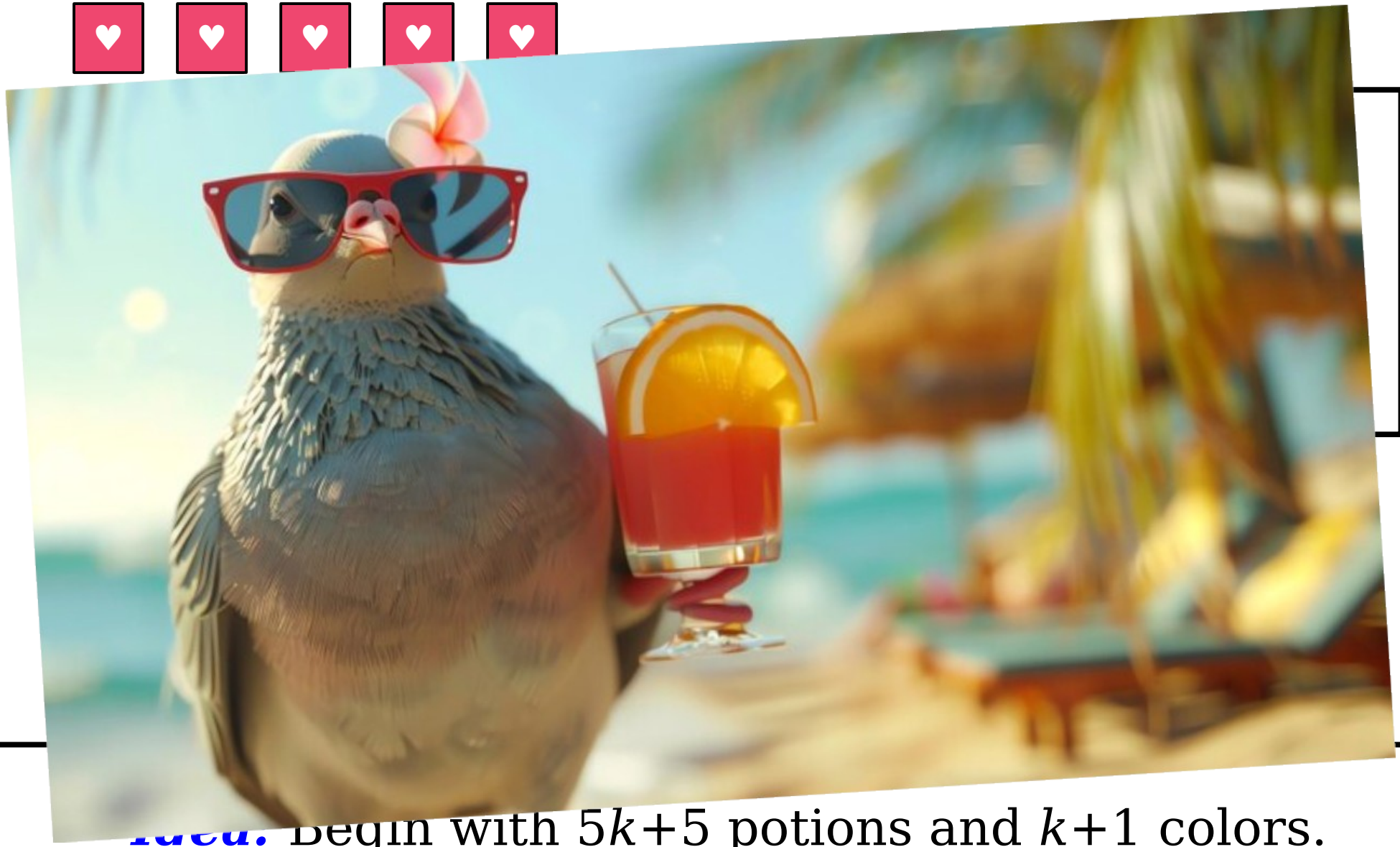

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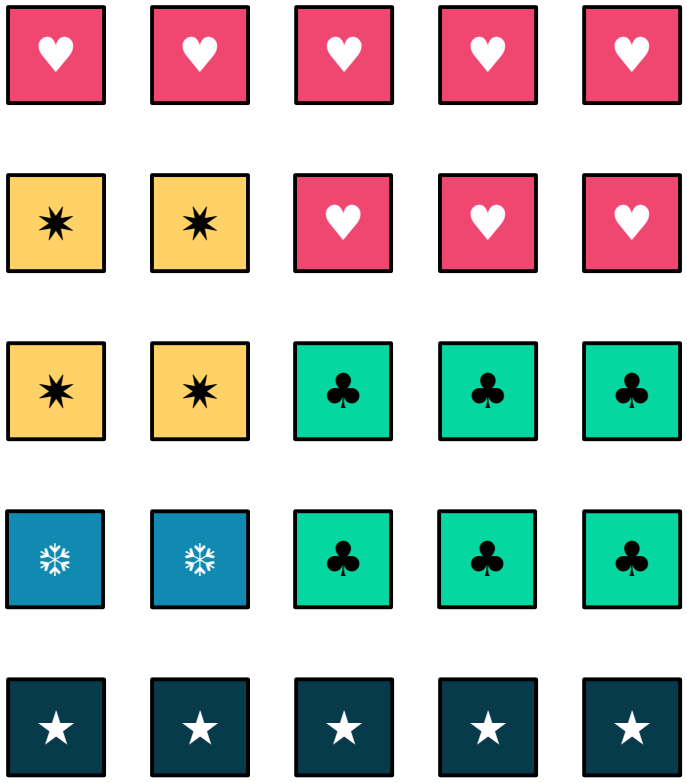

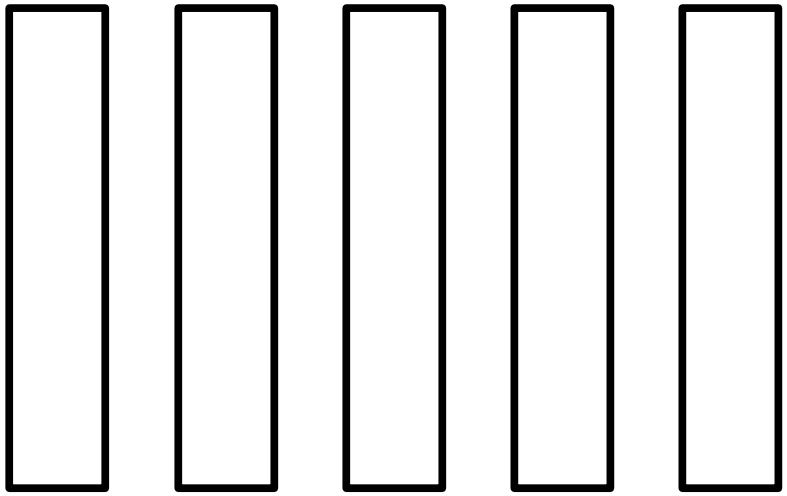

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Theorem: Every group of $5n$ potions of n colors has a good split.

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A nice abbreviation of "generalized pigeonhole principle."

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We need to argue there's some other color that we can use to "top off" the shelf of the purple potions. How do we do that?

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In each case, we form a shelf with 5 potions of at most two different colors and are left with $5k$ potions of k colors.

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In each case, we form a shelf with 5 potions of at most two different colors and are left with $5k$ potions of k colors. By our IH, there is a good split for the remaining potions.

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As our base case, we prove $P(0)$, that any group of 0 potions of 0 colors has a good split. Pick any group of 0 potions. Placing them onto 0 shelves satisfies the requirement of a good split, so $P(0)$ holds.

For our inductive step, pick some $k \in \mathbb{N}$ and assume $P(k)$ holds: any group of $5k$ potions of k colors has a good split. We will prove $P(k+1)$: that any group of $5k+5$ potions of $k+1$ colors has a good split.

Pick any group of $5k+5$ potions of $k+1$ colors. By the GPHP, there is a color (call it **purple**) with $p \leq 5$ potions. We consider two cases:

Case 1: $p = 5$. Place all five purple potions onto their own shelf.

- *Case 2:* $p < 5$. By the GPHP, there is some other color (call it **green**) with $g \geq 5$ potions. Place all p purple potions and $5 - p \leq g$ green potions onto one shelf.

In each case, we form a shelf with 5 potions of at most two different colors and are left with $5k$ potions of k colors. By our IH, there is a good split for the remaining potions. That, plus our original shelf, is a good split of the $5k+5$ potions. Thus $P(k+1)$ holds, completing the induction. ■

A Neat Application

- This result on colored potions forms the basis for the ***alias method***, a fast algorithm for simulated rolls of a loaded die in software.
- This in turn has applications throughout computer science.
- Want to learn more? Check out ***this blog post***, which shows how to apply this result.

Mathematical Induction

Part 2

1. Recap from Last Time
2. **The Colored Cubes Problem**
3. “Building Up” vs. “Building Down”
4. Announcements
5. Complete Induction
6. Eating a Chocolate Bar
7. Induction vs. Complete Induction
8. An Important Milestone
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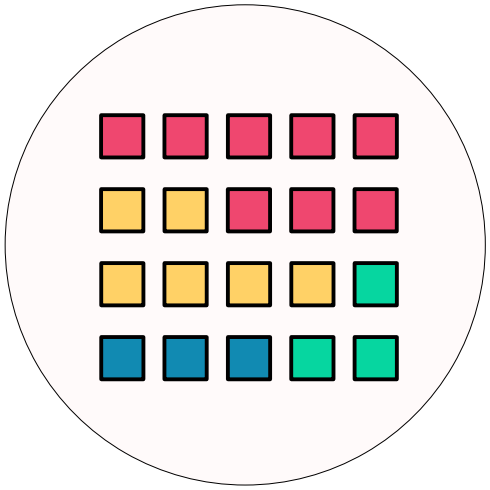
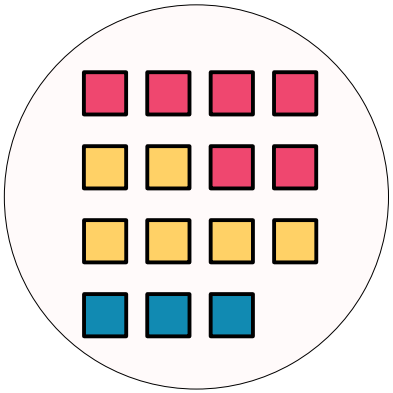
for any group of $5n$ potions of n colors,
there is a good split of those potions.



Start with more potions



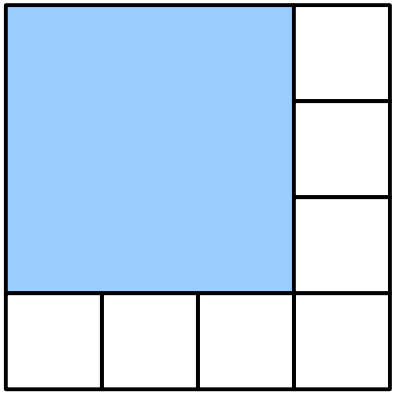
Get to fewer potions

universal quantifier

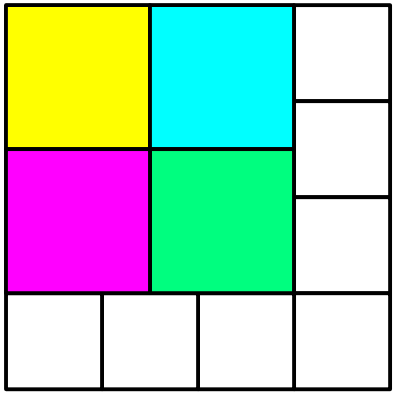
$\forall$

"build down"

"***there exists*** a way to subdivide
a square into n squares."



Start with fewer squares



Get to more squares

existential quantifier

$\exists$

"build up"

Following the Rules

- When working with square subdivisions, our predicate looked like this:

  $P(n)$ is "***there exists*** a way to subdivide a square into n squares."

- When working with colored potions, our predicate looked like this:

  $P(n)$ is "***for any*** group of $5n$ potions of n colors, there is a good split of those potions."

- With squares, the quantifier is $\exists$. With potions, the first quantifier is $\forall$.

- This fundamentally changes the "feel" of induction.

Build Up with ∃

- In the case of squares, in our inductive step, we prove

  If

  there exists a subdivision into k squares,

  then

  there exists a subdivision into $k+3$ squares.

- Assuming the antecedent gives us a concrete subdivision into k squares.

- Proving the consequent means finding some way to subdivide in to $k+3$ squares.

- The inductive step goal is to "***build up***:" start with a smaller number of squares, and somehow work out what to do to get a larger number of squares.

Build Down with ∀

- In the colored potions case, in our inductive step, we prove

  If

  for all groups of $5k$ potions of k colors,
  there's a good split

  then

  for all groups of $5k+5$ potions of $k+1$ colors,
  there's a good split

- Assuming the antecedent means once we find $5k$ potions and k colors, we can group them into a good split.

- Proving the consequent means picking an arbitrary group of $5k+5$ potions of $k+1$ colors and looking for a good split.

- The inductive step goal is to "***build down***:" start with a larger set of potions, then find a way to turn it into a smaller set of potions.

Some Notes

- Not all predicates $P(n)$ will have the form outlined here.
  - That's okay! Just use the normal rules for assuming and proving things.
  - Think of these as quick shorthands rather than fundamentally new strategies.
- In all cases, assume $P(k)$ and prove $P(k+1)$.
  - All that changes is what you do to assume $P(k)$ and what you do to prove $P(k+1)$.
- When in doubt, consult the assume/prove table.
  - It really does work for all cases!

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Problem Sets

- PS4 was due at 1:00 PM.
  - As always, feel free to use one of your three free late days to submit by 1:00 PM on Saturday if needed.
- PS5 will be posted after class and is due next Friday at 1:00 PM.
  - You know the drill: ask questions on Ed or office hours if you have them. That’s what we’re here for!

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First, an interlude!

Interlude:

Rule of Product

and

Rule of Sum!

Rule of Product

- Suppose you want to complete some tasks that involves three independent steps: you do A, then you do B, and then C.
- Suppose there are x ways to do A, y ways to do B, and z ways to do C.
- **Example:** ordering a drink, sandwich, and bakery treat at a cafe.
- How many ways are there to complete this task?
- **Answer:** xyz

From the Rule of Product

Rule of Sum

- Suppose you want to complete some tasks that involves doing *exactly one* of A or B or C.
- Suppose there are x ways to do A, y ways to do B, and z ways to do C.
- **Example:** limiting yourself to ordering exactly one: a drink, a sandwich, or a bakery treat at a cafe.
- How many ways are there to complete this task?
- **Answer:** $x + y + z$

From the Rule of Sum

Mathematical Induction

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First, an interlude!

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If you are the ***leftmost*** person
in your row, stand up right now.

Everyone else: stand up as soon as the
person to your left in your row stands up.

Round Two!

If you are the ***leftmost*** person
in your row, stand up right now.

Everyone else: stand up as soon as
everyone left of you in your row stands up.

What Just Happened?

This is kinda like $P(0)$.

If you are the ***leftmost*** person in your row, stand up right now.

Everyone else: stand up as soon as the person to your left in your row stands up.

This is kinda like $P(k) \to P(k+1)$.

What Just Happened?

This is kinda like $P(0)$.

If you are the ***leftmost*** person in your row, stand up right now.

Everyone else: stand up as soon as ***everyone*** left of you in your row stands up.

What sort of sorcery is this?

Let P be some predicate. The ***principle of complete induction*** states that if

If it starts true…

$P(0)$ is true

and

…and it stays true…

for all $k \in \mathbb{N}$, if $P(0)$, …, and $P(k)$ are true, then $P(k+1)$ is true

then

$\forall n \in \mathbb{N}.\ P(n)$

…then it's always true.

Mathematical Induction

- You can write proofs using the principle of mathematical induction as follows:
  - Define some predicate $P(n)$ to prove by induction on n.
  - Choose and prove a base case (probably, but not always, $P(0)$).
  - Pick an arbitrary $k \in \mathbb{N}$ and assume that $P(k)$ is true.
  - Prove $P(k+1)$.
  - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

Complete Induction

- You can write proofs using the principle of ***complete*** induction as follows:
  - Define some predicate $P(n)$ to prove by induction on n.
  - Choose and prove a base case (probably, but not always, $P(0)$).
  - Pick an arbitrary $k \in \mathbb{N}$ and assume that $\boldsymbol{P(0)}$, $\boldsymbol{P(1)}$, $\boldsymbol{P(2)}$, **...**, **and** $\boldsymbol{P(k)}$ are all true.
  - Prove $P(k+1)$.
  - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

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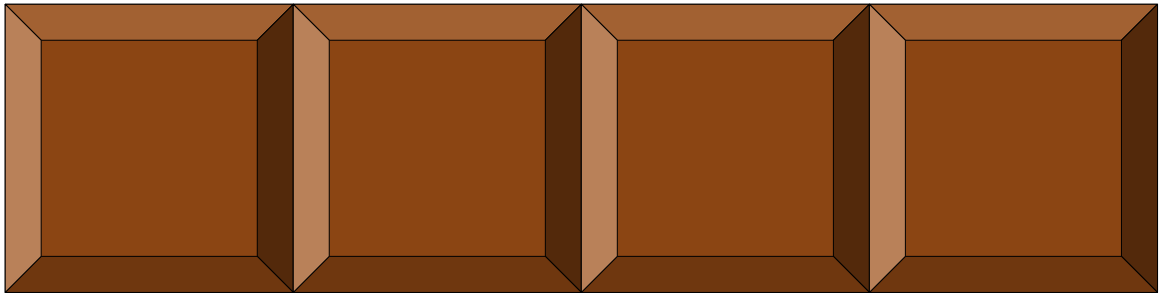
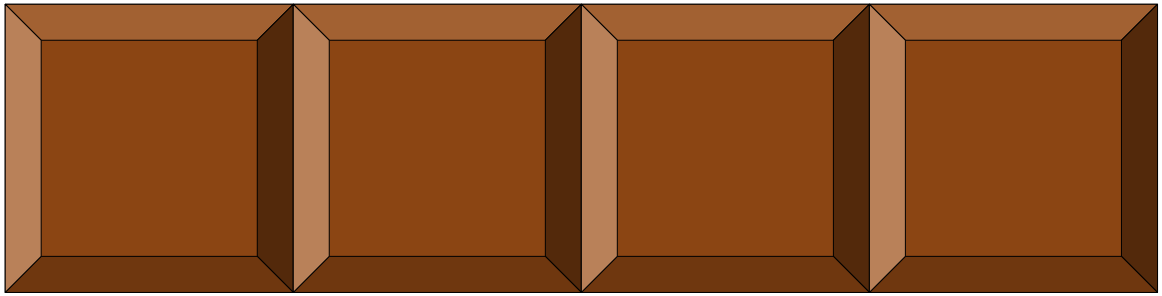
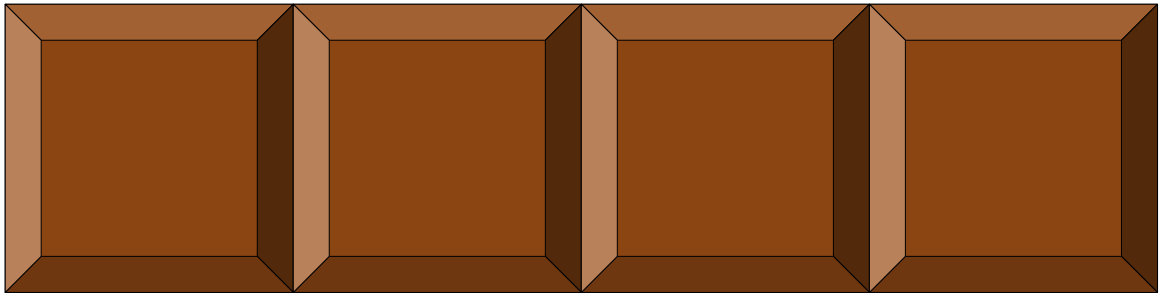


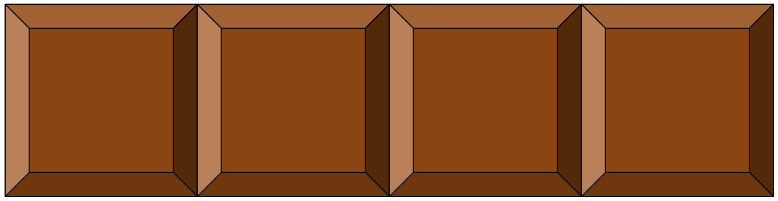


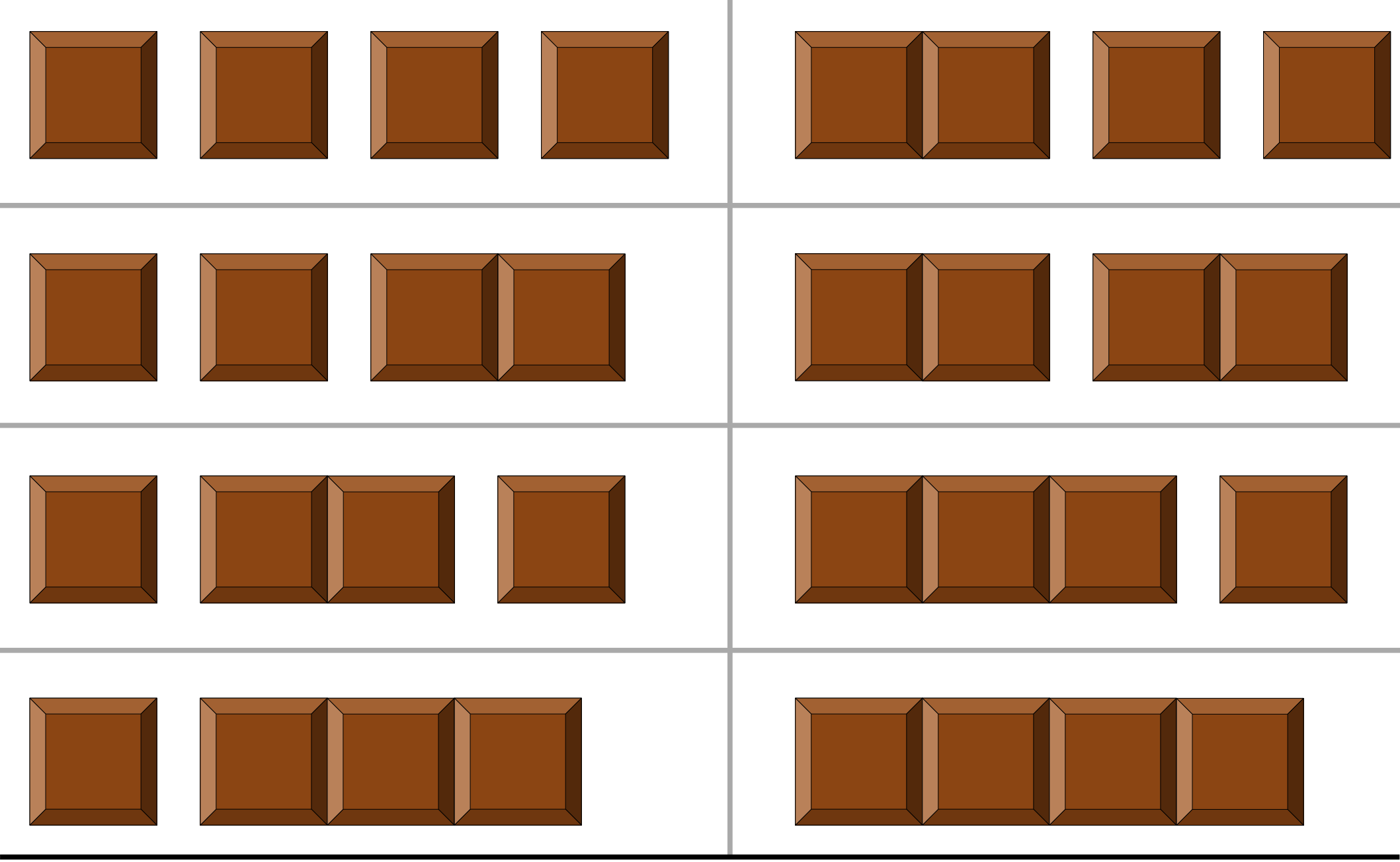


Eating a Chocolate Bar

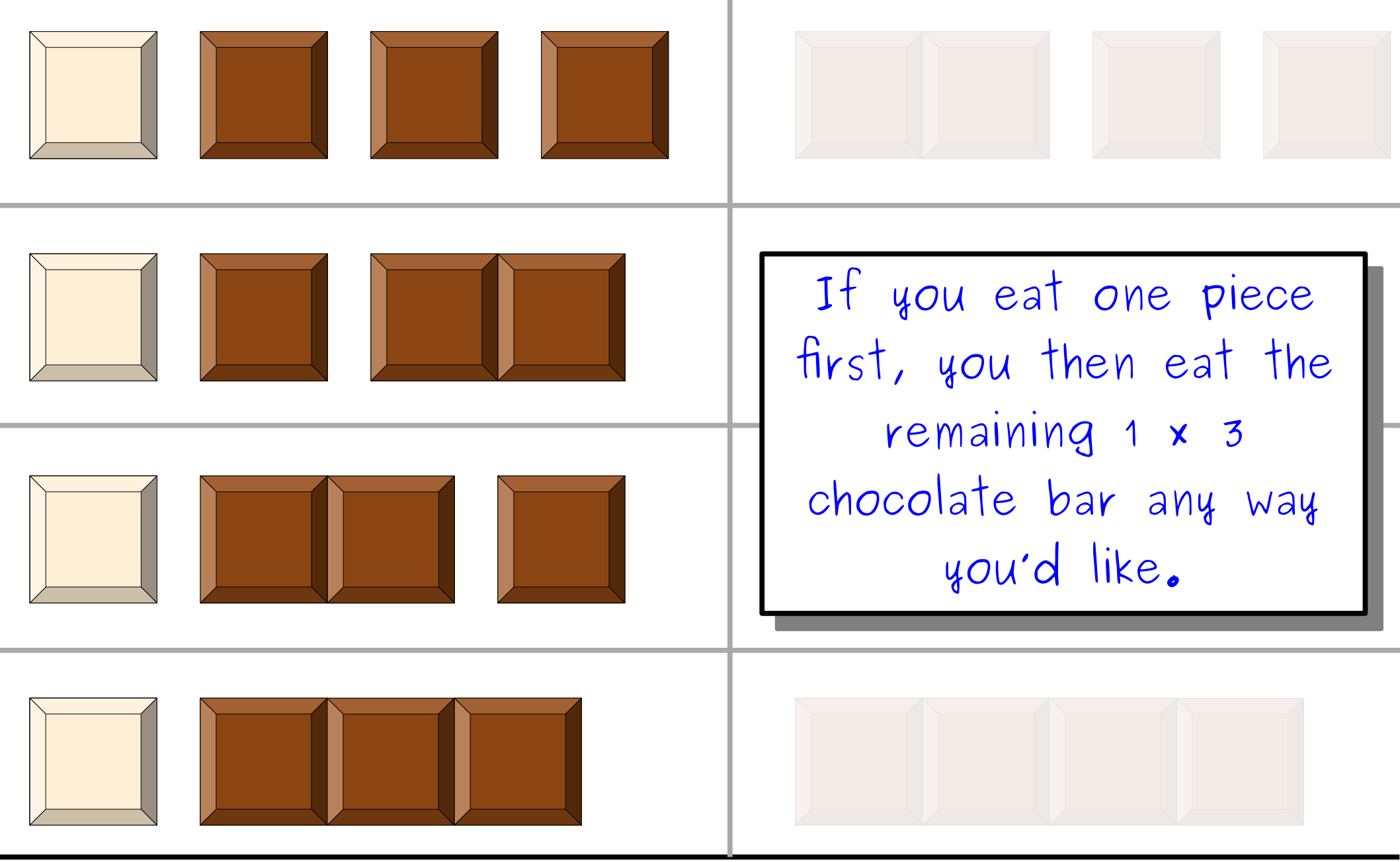
Eating a Chocolate Bar

- You have a $1 \times n$ chocolate bar subdivided into 1×1 squares.
- You eat the chocolate bar from left to right by breaking off one or more squares and eating them in one (possibly enormous) bite.
- How many ways can you eat a…
  - 1×1 chocolate bar?
  - 1×2 chocolate bar?
  - 1×3 chocolate bar?
  - 1×4 chocolate bar?





There are eight ways to eat a 1×4 chocolate bar.



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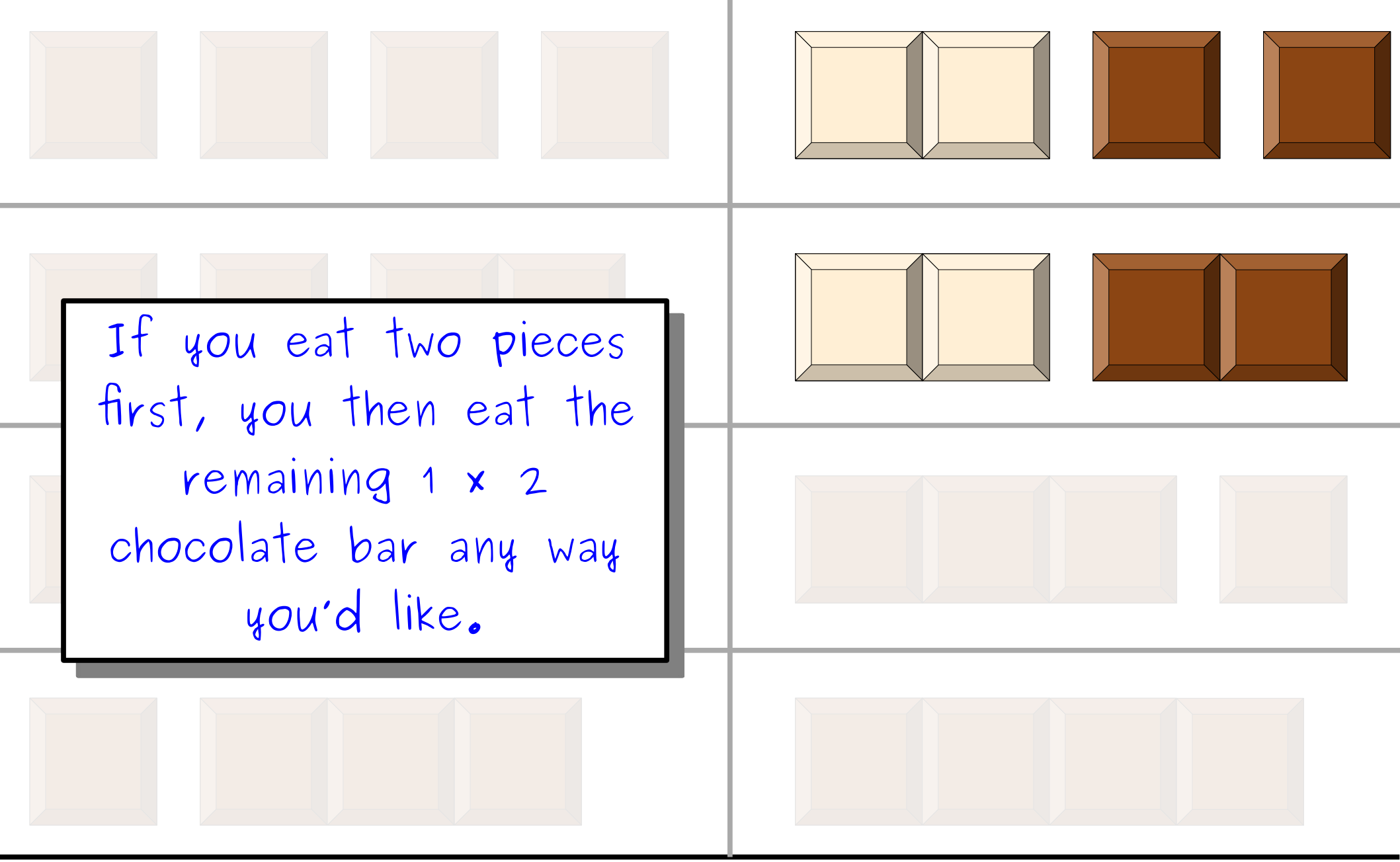

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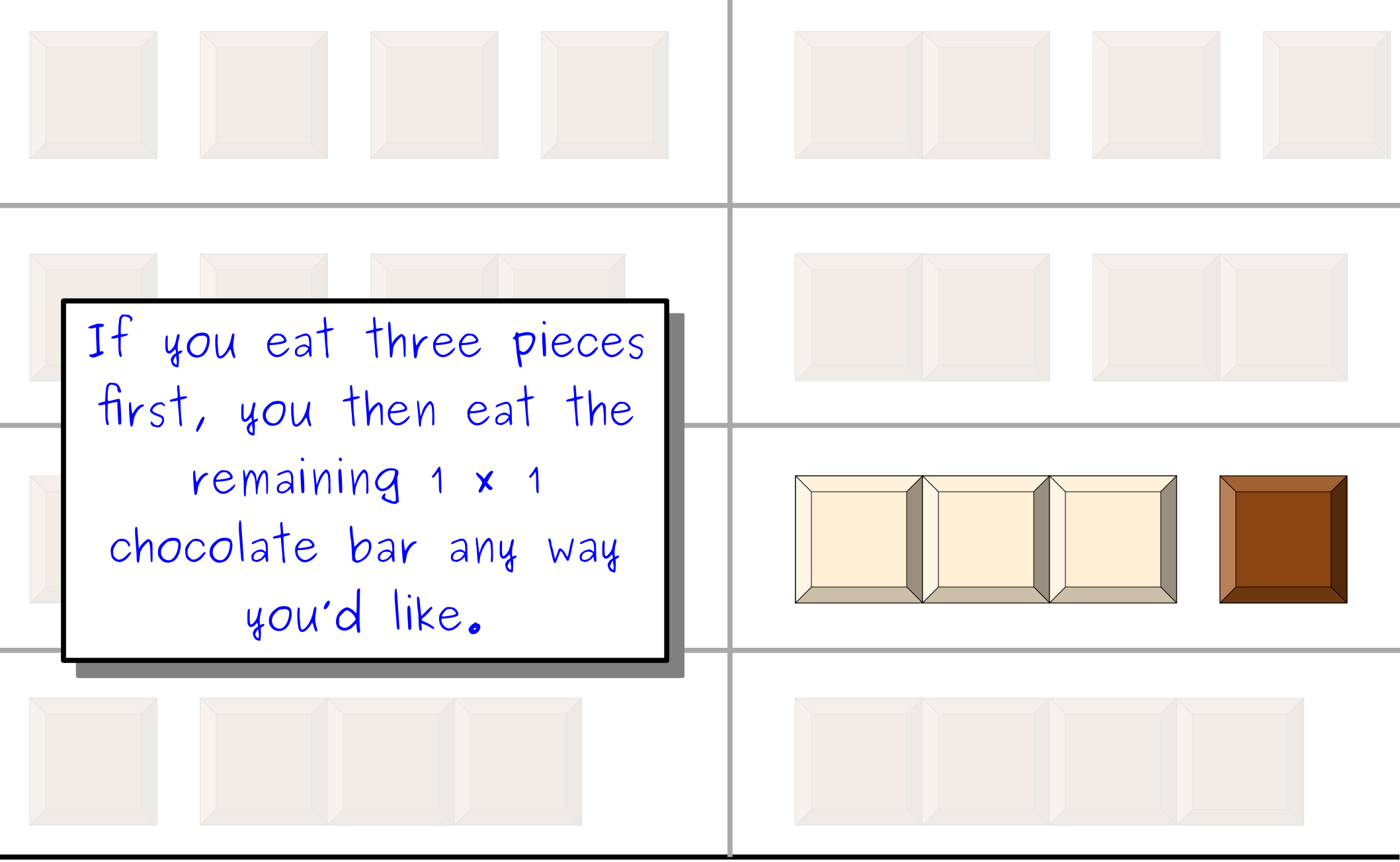
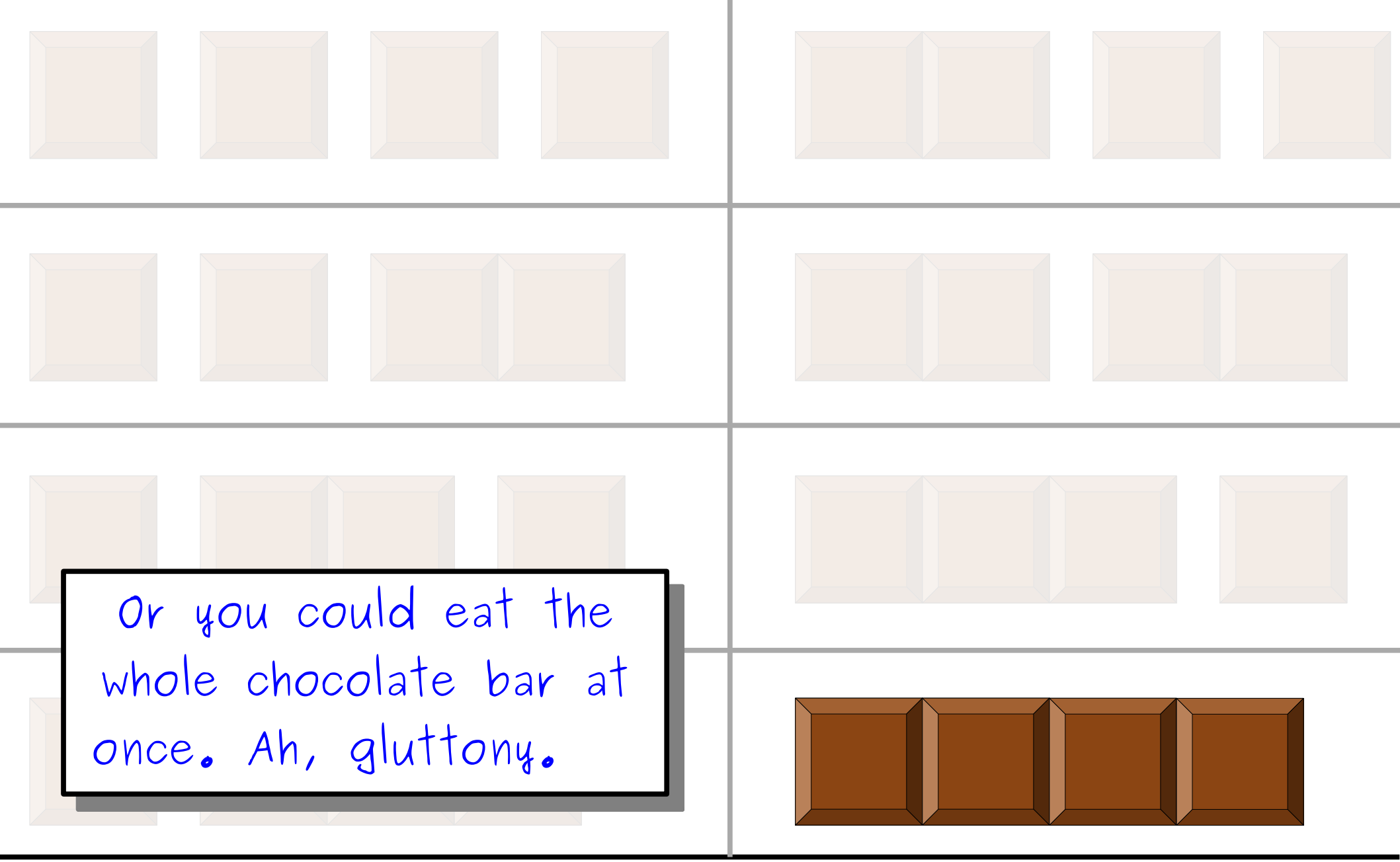


There are eight ways to eat a 1×4 chocolate bar.



There are eight ways to eat a 1×4 chocolate bar.

Eating a Chocolate Bar

- There's…
  - 1 way to eat a 1×1 chocolate bar,
  - 2 ways to eat a 1×2 chocolate bar,
  - 4 ways to eat a 1×3 chocolate bar, and
  - 8 ways to eat a 1×4 chocolate bar.
- ***Our guess:*** There are 2^{n-1} ways to eat a $1 \times n$ chocolate bar for any natural number $n \geq 1$.
- And we think it has something to do with this insight: we eat the bar either by
  - eating the whole thing in one bite, or
  - eating some piece of size k, then eating the remaining $n - k$ pieces however we'd like.
- Let's formalize this!

Theorem: For any natural number $n \geq 1$, the number of ways to eat a $1 \times n$ chocolate bar from left to right is 2^{n-1}.

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Proof: Let $P(n)$ be "the number of ways to eat a $1 \times n$ chocolate bar from left to right is 2^{n-1}." We will prove by induction that $P(n)$ holds for all natural numbers $n \geq 1$, from which the theorem follows.

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As our base case, we prove $P(1)$, that the number of ways to eat a 1×1 chocolate bar from left to right is $2^{1-1} = 1$.

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As our base case, we prove $P(1)$, that the number of ways to eat a 1×1 chocolate bar from left to right is $2^{1-1} = 1$. The only option here is to eat the entire chocolate bar at once, so there's just one way to eat it, as needed.

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For our inductive step, assume for some arbitrary natural number $k \geq 1$ that $P(1)$, ..., and $P(k)$ are true.

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Proof: Let $P(n)$ be "the number of ways to eat a $1 \times n$ chocolate bar from left to right is 2^{n-1}." We will prove by induction that $P(n)$ holds for all natural numbers $n \geq 1$, from which the theorem follows.

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For our inductive step, assume for some arbitrary natural number $k \geq 1$ that $P(1)$, ..., and $P(k)$ are true. We need to show $P(k+1)$ is true, that the number of ways to eat a $1 \times (k+1)$ chocolate bar is 2^k.

Theorem: For any natural number $n \geq 1$, the number of ways to eat a $1 \times n$ chocolate bar from left to right is 2^{n-1}.

Proof: Let $P(n)$ be "the number of ways to eat a $1 \times n$ chocolate bar from left to right is 2^{n-1}." We will prove by induction that $P(n)$ holds for all natural numbers $n \geq 1$, from which the theorem follows.

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There are two options for how to eat the bar.

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There are two options for how to eat the bar. First, we can eat the whole chocolate bar in one bite.

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- There are two options for how to eat the bar. First, we can eat the whole chocolate bar in one bite. Second, we could eat a piece of size r for some $1 \leq r \leq k$, leaving a chocolate bar of size **???**

Theorem: For any natural number $n \geq 1$, the number of ways to eat a $1 \times n$ chocolate bar from left to right is 2^{n-1}.

Proof: Let $P(n)$ be "the number of ways to eat a $1 \times n$ chocolate bar from left to right is 2^{n-1}." We will prove by induction that $P(n)$ holds for all natural numbers $n \geq 1$, from which the theorem follows.

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There are two options for how to eat the bar. First, we can eat the whole chocolate bar in one bite. Second, we could eat a piece of size r for some $1 \leq r \leq k$, leaving a chocolate bar of size $k+1-r$, then eat that chocolate bar from left to right.

Theorem: For any natural number $n \geq 1$, the number of ways to eat a $1 \times n$ chocolate bar from left to right is 2^{n-1}.

Proof: Let $P(n)$ be "the number of ways to eat a $1 \times n$ chocolate bar from left to right is 2^{n-1}." We will prove by induction that $P(n)$ holds for all natural numbers $n \geq 1$, from which the theorem follows.

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This is the spot where we need complete induction. We know we have a smaller chocolate bar, but its size isn't necessarily k.

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Summing up this first option, plus all choices of r for the second option, we see that the number of ways to eat the chocolate bar is

$$1 + 2^0 + 2^1 + \ldots + 2^{k-1}$$

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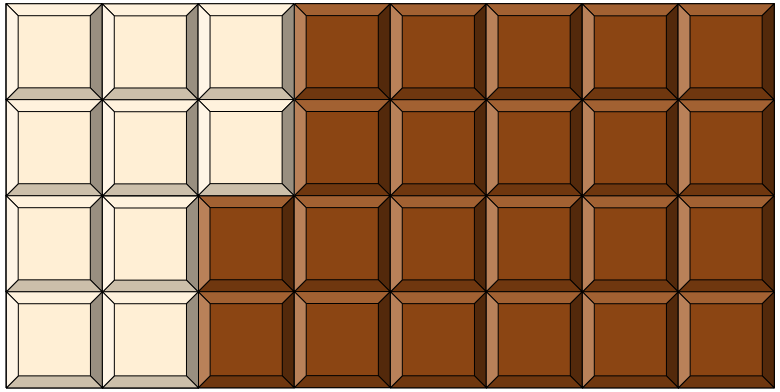
More on Chocolate Bars

- Imagine you have an $m \times n$ chocolate bar. Whenever you eat a square, you have to eat all squares above it and to the left.
- How many ways are there to eat the chocolate bar?



- ***Open Problem:*** Find a non-recursive exact formula for this number, or give an approximation whose error drops to zero as m and n tend toward infinity.

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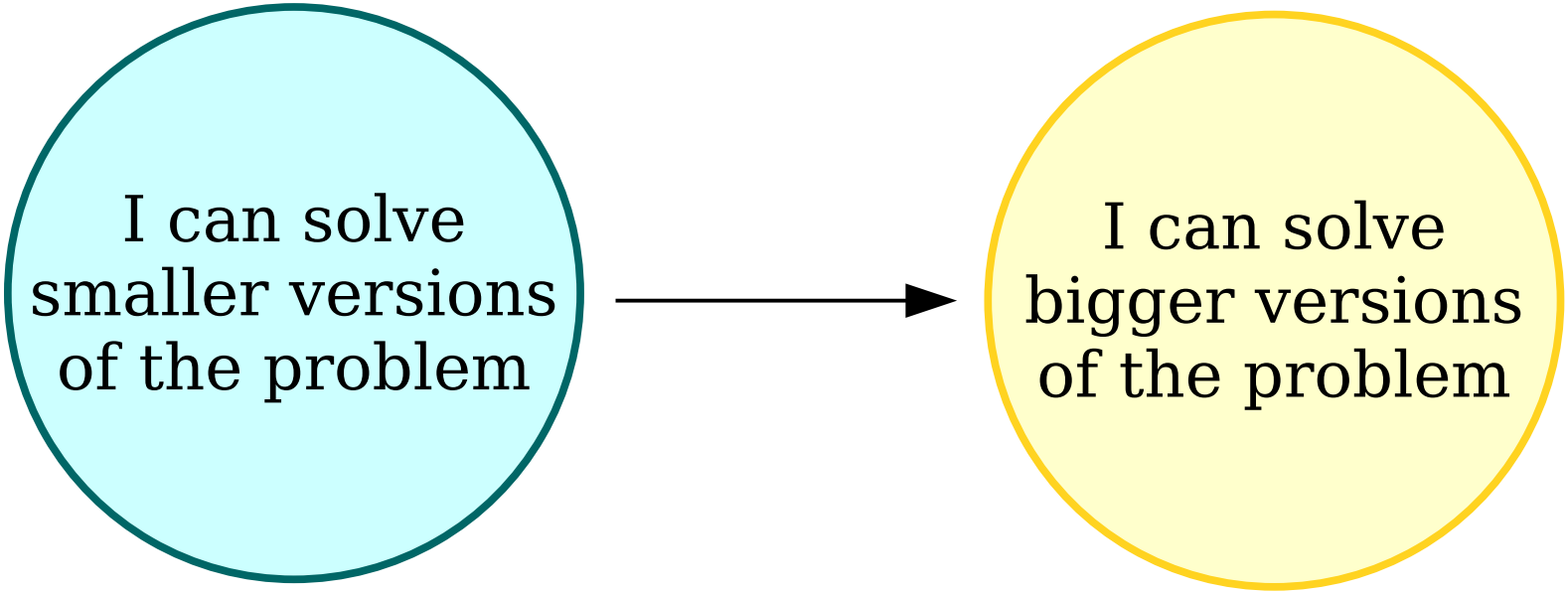

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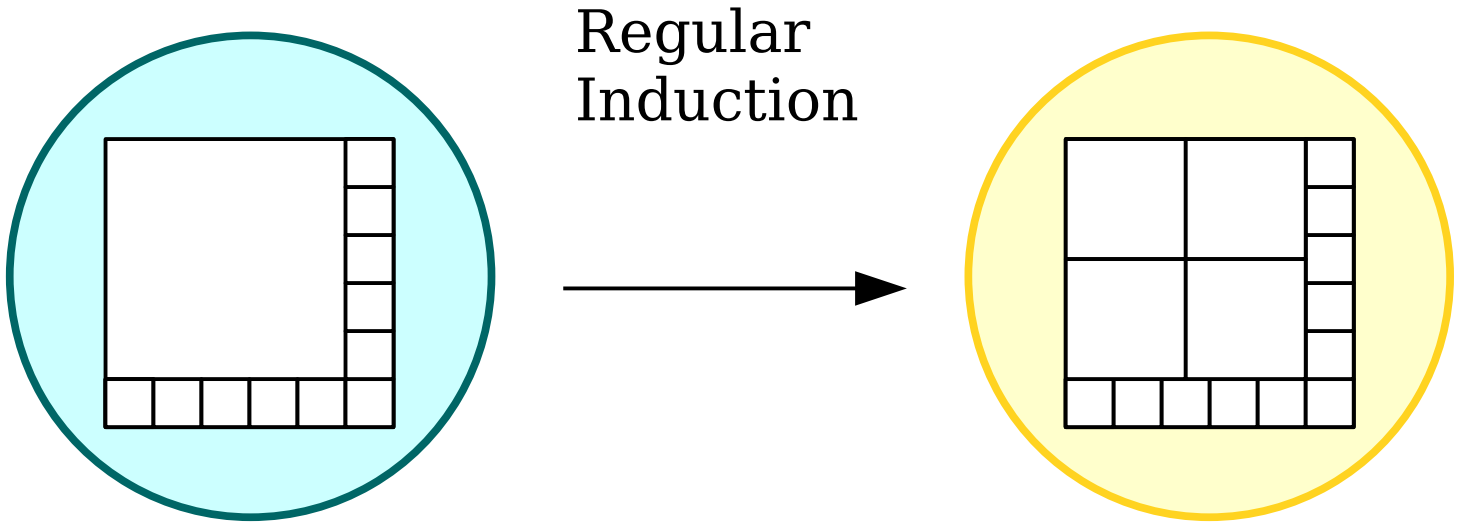

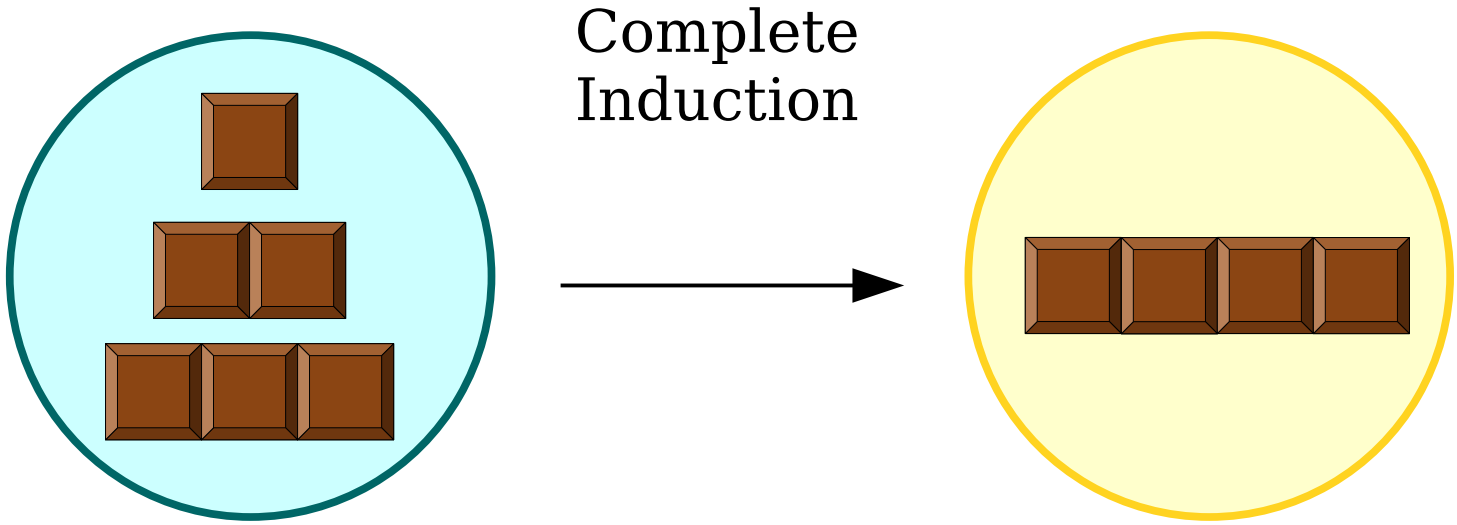

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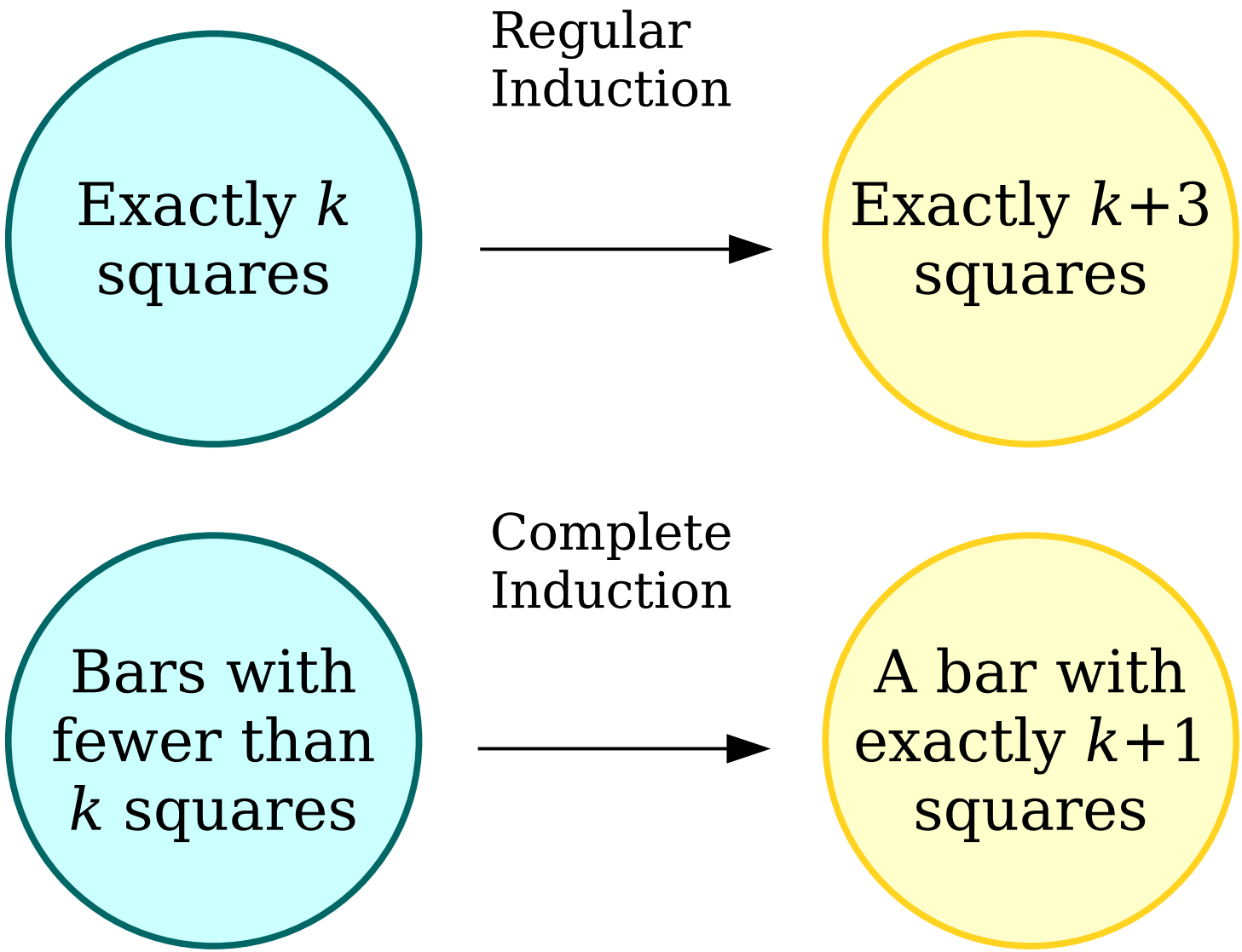

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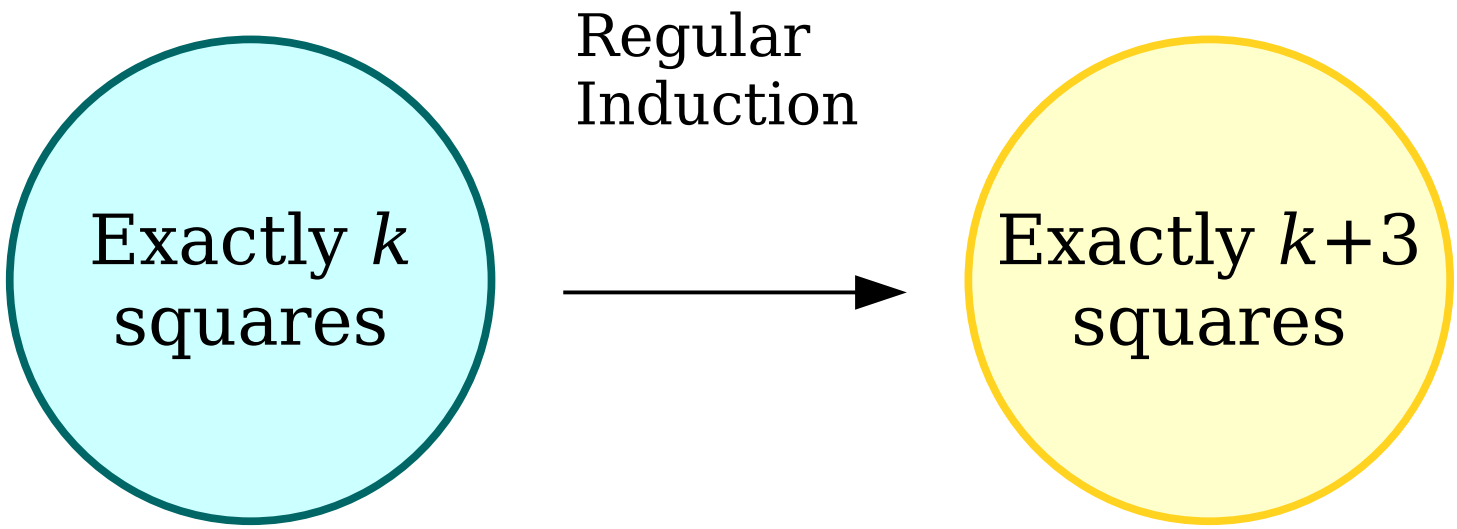

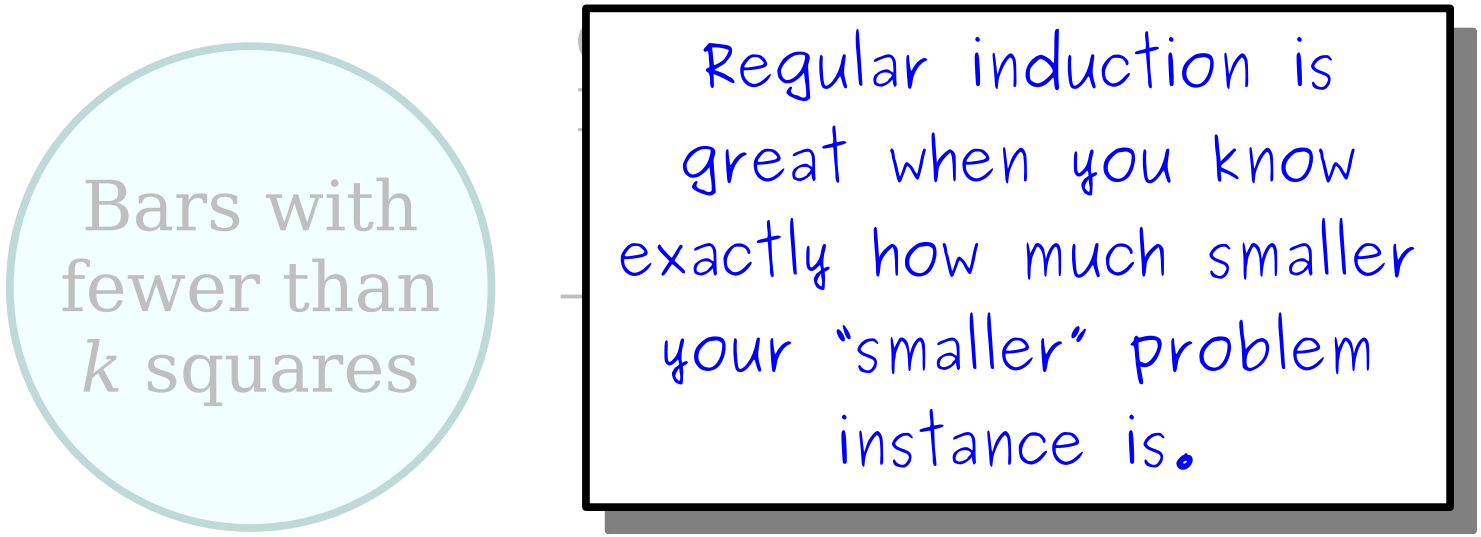

Induction vs. Complete Induction

Complete induction is great when you know things get smaller, but you're not sure by how much.

Regular Induction

Exactly $k+3$ squares

Complete Induction

Bars with fewer than k squares

A bar with exactly $k+1$ squares

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Recap: ***Discrete Mathematics***

- The past five weeks have focused exclusively on discrete mathematics:

Induction	Functions
Graphs	The Pigeonhole Principle
Formal Proofs	Mathematical Logic
Set Theory	

- These are building blocks we will use throughout the rest of the quarter.
- These are building blocks you will use throughout the rest of your CS career.

Next Up: ***Computability Theory***

- It's time to switch gears and address the limits of what can be computed.
- We'll explore these questions:
  - How do we model computation itself?
  - What exactly is a computing device?
  - What problems can be solved by computers?
  - What problems *can't* be solved by computers?
- ***We're about to explore the boundaries of what computers could ever be made to do!***

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Next Time

- ***Formal Language Theory***
  - How are we going to formally model computation?
- ***Finite Automata***
  - A simple but powerful computing device made entirely of math!
- ***DFAs***
  - A fundamental building block in computing.